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Geometry of Orientifolds with NS-NS B-flux

Zurab Kakushadze*
C.N. Yang Institute for Theoretical Physics
State University of New York, Stony Brook, NY 11794
and
Institute for Theoretical Physics
University of California, Santa Barbara, CA 93106
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Abstract

We discuss geometry underlying orientifolds with non-trivial NS-NS B-flux. If D-branes wrap a torus with B-flux the rank of the gauge group is reduced due to non-commuting Wilson lines whose presence is implied by the B-flux. In the case of D-branes transverse to a torus with B-flux the rank reduction is due to a smaller number of D-branes required by tadpole cancellation conditions in the presence of B-flux as some of the orientifold planes now have the opposite orientifold projection. We point out that T-duality in the presence of B-flux is more subtle than in the case with trivial B-flux, and it is precisely consistent with the qualitative difference between the aforementioned two setups. In the case where both types of branes are present, the states in the mixed (*e.g.*, 59) open string sectors come with a non-trivial multiplicity, which we relate to a discrete gauge symmetry due to non-zero B-flux, and construct vertex operators for the the mixed sector states. Using these results we revisit K3 orientifolds with B-flux (where K3 is a $T^4/\mathbf{Z}_M$ orbifold) and point out various subtleties arising in some of these models. For instance, in the $\mathbf{Z}_2$ case the conformal field theory orbifold does not appear to be the consistent background for the corresponding orientifolds with B-flux. This is related to the fact that non-zero B-flux requires the presence of both O5$^-$- as well as O5$^+$-planes at various $\mathbf{Z}_2$ orbifold fixed points, which appears to be inconsistent with the presence of the *twisted* B-flux in the conformal field theory orbifold. We also consider four dimensional $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetric orientifolds. We construct consistent four dimensional models with B-flux which do not suffer from difficulties encountered in the K3 cases.

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*E-mail: zurab@insti.physics.sunysb.edu

I. INTRODUCTION AND SUMMARY

In recent years six and four dimensional orientifolds have been extensively studied, and much progress has been made in understanding such string compactifications. Various $\mathcal{N}=1$ supersymmetric six dimensional orientifold vacua were constructed, for instance, in [1–5]. Generalizations of these constructions to $\mathcal{N}=1$ supersymmetric four dimensional orientifold vacua have also been discussed, for instance, in [6–20].

Most of the aforementioned discussions have been confined to orientifolds with vanishing NS-NS antisymmetric tensor backgrounds. However, generalizations to cases with non-trivial NS-NS B-flux have also been considered. Thus, in [21][1] toroidal Type I compactifications with non-zero B-field were studied. In [21] it was pointed out that even though the NS-NS 2-form is projected out of the closed sting spectrum by the orientifold projection Ω, quantized expectation values of B_{ij} are allowed with i,j corresponding to the toroidally compactified coordinates. In particular, since the components of B_{ij} are defined up to unit shifts $B_{ij} \rightarrow B_{ij}+1$, and B_{ij} is odd under the world-sheet parity reversal Ω, the allowed background values for the components of B_{ij} are 0 and 1/2. Furthermore, in [21] it was found that the rank of the gauge group coming from D9-branes wrapped on tori with non-vanishing half-integer B-flux is reduced from 16 (that is, the rank of the original $SO(32)$ gauge group) down to $16/2^{b/2}$, where b is the rank (which is always even) of the matrix B_{ij}. This rank reduction is evident once one considers the cylinder partition function for D9-branes wrapped on such tori. However, the partition function alone does not provide a clear geometric interpretation of the rank reduction phenomenon. In particular, one might conclude that in the cases with non-zero B-field we have only $32/2^{b/2}$ D9-branes instead of the usual 32 D9-branes. This, however, does not appear to be the case. Thus, as was originally pointed out in [8], the Ω orientifold of Type IIB with non-zero B-flux can be viewed as a toroidal Type I compactification with all 32 D9-branes in the presence of *non-commuting* Wilson lines. This point was further elaborated in more detail in [23]. For the reasons that will become clear in a moment, in this paper we will review the approach of [8,23] in detail.

Further progress in understanding toroidal orientifolds with non-zero B-flux was made in [24], where D-branes transverse to tori with non-trivial B_{ij} were considered. In particular, it was argued in [24] that if we, say, consider O7-planes transverse to a 2-torus T^2 with $B_{12}=1/2$ in the directions of T^2, then we have two types of orientifold planes O7$^-$ and O7$^+$, where the O7$^-$-plane refers to the usual orientifold plane with the SO type of orientifold projection on the Chan-Paton charges, whereas the O7$^+$-plane refers to the orientifold plane with the Sp type of projection. More concretely, out of the four O7-planes (located at the 4 points on T^2 fixed under the reflection $R: X_{1,2} \rightarrow -X_{1,2}$, where R is now a part of the orientifold projection $\Omega R(-1)^{F_L}$) three are of the O7$^-$ type, while one is of the O7$^+$ type (to be contrasted with the case with trivial B-flux where all four O7-planes are of the O7$^-$ type). The R-R charges of the O7$^-$- and O7$^+$-planes are -8 and $+8$, respectively, so that the total R-R charge to be canceled by D7-branes is -16. This implies that we must introduce 16 (instead of 32) D7-branes in the presence of non-zero B_{12}, hence "rank reduction". However,

[1] For a more recent discussion of toroidal Type I compactifications, see [22].

as we will point out in the following, the mechanisms for the rank reduction in the cases of D-branes wrapped on tori with non-zero B_{ij} *vs.* D-branes transverse to such tori are *different*. In the former case we still have 32 D-branes, and the rank reduction occurs due to non-commuting Wilson lines [8,23]. In the latter case the number of D-branes is not 32 but $32/2^{b/2}$ due to the fact that some of the orientifold planes are no longer of the SO but Sp type. Thus, in this case there is really no "rank reduction". Moreover, the two cases are *not* T-dual to each in the usual sense as T-duality is more subtle in the presence of the B-flux (some aspects of T-duality in this context were discussed in [24]). We will make this more precise in the following using the approach of [8,23], which, in particular, makes it clear that one of the O7-planes must indeed be of the Sp type.

So far we have mentioned toroidal orientifold compactifications which preserve 16 supersymmetries. To obtain backgrounds with reduced supersymmetry we need to consider compactifications with non-trivial holonomy. A step in this direction was made in [5], where K3 orientifolds with non-zero B-field were discussed. More concretely, in [5] the following backgrounds were considered. Start with Type IIB on K3 $= T^4/\mathbf{Z}_M$, where $M = 2, 3, 4, 6$ so that the orbifold action on T^4 is crystallographic. Then consider the Ω orientifold, where Ω acts not as in the smooth case[2] but as in [3], that is, Ω maps the g^k twisted sector to the g^{M-k} twisted sector [26], where g is the generator of the orbifold group $\mathbf{Z}_M$, and $k = 0, \dots, M-1$. Let us assume that there is a non-zero half-integer B-flux of rank b in the directions of K3. This is the setup of [5]. *A priori* we expect that in the $\mathbf{Z}_3$ case we have no D5-branes (for the above choice of the orientifold projection), but some number of D9-branes. In the other three cases, namely, $\mathbf{Z}_2, \mathbf{Z}_4, \mathbf{Z}_6$, we expect both D9- and D5-branes to be present. The aforementioned question of how many D9-branes we have in these backgrounds, namely, whether we have 32 or $32/2^{b/2}$ D9-branes, becomes extremely relevant, at least in the cases where we have both D9- and D5-branes. The reason for this is the following. As was originally pointed out in [5], the 59 sector states arise with multiplicity of $2^{b/2}$, which is 1 in the absence of the B-field, but becomes non-trivial whenever the B-field is non-zero. This fact becomes evident, as was explained in [5], if one examines the boundary states for the D9- and D5-branes in the presence of the B-field[3]. In particular, this multiplicity of states is in accord with the tadpole/anomaly cancellation requirements in, say, the $\mathbf{Z}_2$ model [5]. However, just by looking at the boundary states (or, equivalently, the partition function), the geometric interpretation of this multiplicity of states in the 59 sector is obscure. In particular, in string theory we expect that no two states should have identical vertex operators. Thus, one should be able to distinguish the otherwise degenerate 59 states (in the presence of the B-field) from each other by some quantum numbers. In this paper we will give an explicit answer to this question. In particular, we will use the approach of [8,23], and point out that the number of D5-branes (if present) is $32/2^{b/2}$, whereas the number of D9-branes is always 32, albeit the rank of the 99 gauge group is $16/2^{b/2}$ (for the reasons mentioned above). The 59 sector degeneracy then is related to the fact that in these sectors

[2]Orientifolds of Type IIB on smooth K3 surfaces with non-zero B-flux were discussed in [25].

[3]Equivalently, one can examine the annulus partition function, and see that the multiplicity of states in the 59 sector is indeed $2^{b/2}$. This was done for the $\mathbf{Z}_2$ models in [27].

there are $2^{b/2}$ different vertex operators with otherwise identical quantum numbers, and the former are distinguished by $2^{b/2}$ Chan-Paton degrees of freedom that 32 D9-branes have on top of those corresponding to the unbroken gauge group. In particular, as we will see in the following, these degrees of freedom correspond to $2^{b/2}$ different charges carried by the 59 sector states under a discrete gauge symmetry, namely, $(\mathbf{Z}_2)^{\otimes(b/2)}$, arising in the 99 sector. In fact, one can turn this point around and argue that the multiplicity of states in the 59 sector gives us a hint for the number of D9-branes being 32 and not $32/2^{b/2}$.

Understanding the geometry underlying toroidal orientifolds with NS-NS B-flux, which is one of the aims of this paper, is a useful tool for shedding light on other non-trivial orientifold compactifications with B-flux, say, with reduced number of supersymmetries. In particular, in this paper we will revisit the K3 orientifolds with B-flux originally discussed in [5]. It turns out that there are various subtleties arising in these models. For instance, as we will argue in the following, the conformal field theory $T^4/\mathbf{Z}_2$ orbifold does not appear to be a consistent background for the corresponding orientifolds with B-flux. This is related to the fact that in the presence of non-trivial B-flux we must have both $\mathrm{O5}^-$- as well as $\mathrm{O5}^+$-planes at various $\mathbf{Z}_2$ orbifold fixed points, and this is incompatible with the presence of the *twisted* B-flux in the conformal field theory orbifold. The $\mathbf{Z}_6$ as well as $\mathbf{Z}_4$ models with B-flux also suffer from this problem. In fact, we will discuss other subtleties arising in the $\mathbf{Z}_6$ and $\mathbf{Z}_4$ models, which appear to be related, just as it turns out to be the case for their $\mathbf{Z}_2$ counterparts, to the fact that here we have 59 sectors whose presence appears to be incompatible with the lack of vector structure dictated by non-zero B-flux. On the other hand, $\mathbf{Z}_3$ models with, say, D5-branes only (but no mixed 59 sector) appear to be consistent[4].

The key points of the above discussion carry over to analogous compactifications on Calabi-Yau orbifolds. In particular, every time we have, say, a $\mathbf{Z}_3$ twist acting in the compact directions with non-zero B-flux, extra care is needed in making sure that tadpole cancellation is compatible with geometric constraints. We illustrate these issues by revisiting the four dimensional models of [14,15], in some of which we also observe various subtleties.

The remainder of this paper is organized as follows. In section II we discuss toroidal orientifolds with B-flux. Our discussion here is essentially a generalization of non-commutative toroidal compactifications in the presence of orientifold planes. This, generalization, however, is *not* completely straightforward as including orientifold planes brings in additional subtle issues. In section III we revisit K3 orientifolds with B-flux, and explain in detail the aforementioned subtle inconsistencies arising in some of these backgrounds. In section IV we discuss four dimensional cases with $SU(3)$ as well $SU(2)$ holonomy. In particular, here we construct consistent orientifold models with B-flux which do not suffer from difficulties encountered in the K3 cases. We comment on various issues in section V.

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[4]Here we should point out that the other aforementioned models might be consistent in some other sense, but we fail to find consistent constructions for them within the (perhaps limited) orientifold framework.

II. TOROIDAL ORIENTIFOLDS WITH B-FLUX

In this section we will review the effects of the B-field in toroidal orientifolds. We will use the approach of [8,23], where the geometric interpretation of such backgrounds is evident. In subsection A we will discuss the cases where D-branes (and the corresponding orientifold planes) wrap a torus with non-zero B-field. In subsection B will will consider the cases where D-branes are transverse to such tori. In subsection C we will generalize our discussion to the cases where both types of D-branes are present. There we will also discuss in detail T-duality in orientifolds with B-flux, and, in particular, argue that the results of this section are consistent with T-duality considerations.

A. D-branes and O-planes Wrapped on Tori with B-flux

Consider Type IIB (it is straightforward to generalize our discussion to Type IIA) on $\mathbf{R}^{1,9-d}\otimes T^d$ in the presence of some number n of Dp-branes completely wrapping the d-torus ($p\geq d$). We would like to study the effect of turning on a quantized (half-integer) B-field in the directions of T^d. For the sake of clarity we will consider D9-branes wrapping T^2. Generalizations to other cases are completely straightforward.

Thus, let us start from Type IIB on $\mathbf{R}^{1,7}\otimes T^2$ in the presence of some number n of D9-branes. Let us first assume the the B-field in the compactified directions is zero: $B_{12}=0$. Let g_{ij} be the metric on T^2. Then in the suitable normalization the left- and right-moving closed string momenta are given by

$$P_{L,R}=\frac{1}{2}\widetilde{e}^i m_i \pm e_i n^i \ , \tag{1}$$

where e_i are two-component vielbeins satisfying $e_i\cdot e_j=g_{ij}$, while $\widetilde{e}^i$ are their duals: $\widetilde{e}^i\cdot\widetilde{e}^j=\widetilde{g}^{ij}$, where $\widetilde{g}^{ij}$ is the inverse of g_{ij}. Also, the integers m_i and n^i are the momentum and winding numbers, respectively. As usual, T^2 can be viewed as a quotient $\mathbf{R}^2/\Lambda$, where the lattice $\Lambda\equiv\{e_i n^i\}$, and the coordinates X_i on T^2 are identified via $X_i\sim X_i+e_i$.

Next, consider a freely acting $\mathbf{Z}_2\otimes\mathbf{Z}_2$ orbifold of this theory defined as follows. Let S_i be a half-lattice shift in the X_i direction, that is, $S_iX_i=X_i+e_i/2$. Note that the set $\{I,S_1,S_2,S_3\}$ forms a freely acting orbifold group isomorphic to $\mathbf{Z}_2\otimes\mathbf{Z}_2$. Here I is the identity element, and $S_3\equiv S_1S_2$. Thus, in this orbifold we have the untwisted sector labeled by I, and three twisted sectors labeled by S_1,S_2,S_3. The left- and right-moving momenta can now be written as

$$P_{L,R}(\alpha^1,\alpha^2)=\frac{1}{2}\widetilde{e}^i m_i\pm e_i(n^i+\frac{1}{2}\alpha^i) \ , \tag{2}$$

where $(\alpha^1,\alpha^2)=(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ in the untwisted, S_1, S_2 and S_3 twisted sectors, respectively. The most general action of S_i on the left- and right-moving momenta (compatible with modular invariance) is given by

$$S_i|P_L,P_R\rangle_{(\alpha^1,\alpha^2)}=\epsilon_i(\alpha^1,\alpha^2)\exp(\pi i m_i)|P_L,P_R\rangle_{(\alpha^1,\alpha^2)} \ , \tag{3}$$

where $\epsilon_i(0,0)\equiv 1$, $\epsilon_1(1,0)=\epsilon_2(0,1)=1$, $\epsilon_1(0,1)=\epsilon_2(1,0)=\epsilon_{1,2}(1,1)=\epsilon$. Here ϵ can take two values: ±1. If $\epsilon=+1$, then we have the usual freely acting $\mathbf{Z}_2\otimes\mathbf{Z}_2$ orbifold. It

is straightforward to show that the resulting theory corresponds to a compactification on $(T^2)'$ with the metric $g'_{ij} = g_{ij}/4$, and zero B-field $B_{12} = 0$. However, if $\epsilon = -1$, in which case we have *discrete torsion* between the generators S_1 and S_2 of the two $\mathbf{Z}_2$ subgroups of the orbifold group (that is, S_1 acts with an extra minus sign in the S_2 twisted sector, and, consequently, S_2 acts with the extra minus sign in the S_1 twisted sector; both S_1 and S_2 act with an extra minus sign in the S_3 twisted sector), then it is straightforward to show that the resulting theory corresponds to a compactification on $(T^2)'$ with the metric $g'_{ij} = g_{ij}/4$, but now the B-field is *non-zero*: $B_{12} = 1/2$. In deriving these results it is important to note that (3) implies that only *even* momentum numbers m_i survive the orbifold projection in all four sectors if there is no discrete torsion (that is, $\epsilon = 1$), while in the case with discrete torsion (that is, $\epsilon = -1$) the momenta kept after the orbifold projection are given as follows. Both m_1 and m_2 are even in the untwisted sector. In the S_1 twisted sector m_1 is even and m_2 is odd. In the S_2 twisted sector m_1 is odd and m_2 is even. Finally, in the S_3 twisted sector both m_1 and m_2 are odd. Taking all of this into account, we can see from (2) and (3) that the resulting left- and right-moving momenta are given by

$$P_{L,R} = \frac{1}{2}\tilde{e}'^i(m'_i - B_{ij}n'^j) \pm e'_i n'^i \ , \tag{4}$$

where the new momentum and winding numbers m'_i and n'^i are now arbitrary integers, the new vielbeins e'_i and their duals $\tilde{e}'^i$ are related to the original ones via $e'_i = e_i/2$, $\tilde{e}'^i = 2\tilde{e}^i$, and the B-field is zero for $\epsilon = 1$, while $B_{12} = 1/2$ for $\epsilon = -1$.

Thus, what we have learned from the above discussion is that we can describe a compactification on $(T^2)'$ with half-integer B-field as a freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold that involves half-shifts along the two cycles of T^2 (the metric on $(T^2)'$ is four times as small as that on T^2, that is, both of the cycles on $(T^2)'$ are half the size of the corresponding cycles on T^2) with discrete torsion between the generators S_1 and S_2 of the two $\mathbf{Z}_2$ subgroups of the orbifold group $\mathbf{Z}_2 \otimes \mathbf{Z}_2$. This approach proves to be convenient in the context of D-branes (and O-planes) wrapping tori with non-zero B-field as we can recast the latter problem into the corresponding freely acting orbifold of a setup where D-branes are wrapping a torus with *zero* B-field. In particular, as we will see in a moment, this provides a geometrization of the cases where D-branes wrap tori with non-zero B-field.

Let us now see what happens to open strings stretched between n D9-branes wrapped on $(T^2)'$. As we have already mentioned, to study this system we can start from n D9-branes wrapped on T^2 without the B-field, and then consider the aforementioned freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold. Thus, before orbifolding we have open string states which can be obtained by the corresponding Kaluza-Klein (KK) compactification of the ten dimensional open string spectrum on T^2. In particular, among other quantum numbers (corresponding to the open string oscillator modes) we have the Kaluza-Klein momenta $m_i \in \mathbf{Z}$ with the contributions to the masses of the corresponding open string states given by $M^2_{\mathbf{m}} = \frac{1}{2}P^2_{\mathbf{m}}$, where $P_{\mathbf{m}} \equiv \tilde{e}^i m_i$ (here $\mathbf{m} \equiv (m_1, m_2)$). The generators S_1 and S_2 of the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold group act on the momentum states as follows:

$$S_i|P_{\mathbf{m}}\rangle = \exp(\pi i m_i)|P_{\mathbf{m}}\rangle \ . \tag{5}$$

However, we must also specify the action of the orbifold group elements on the Chan-Paton charges of D9-branes. It is described by $n \times n$ matrices that form a representation of $\mathbf{Z}_2 \otimes \mathbf{Z}_2$.

Thus, we are free to choose the Chan-Paton matrix γ_I corresponding to the identity element I of the orbifold group as $\gamma_I = I_n$, where here and in the following I_m will denote the $m \times m$ identity matrix. As to the twisted Chan-Paton matrices γ_{S_1} and γ_{S_2} (as well as γ_{S_3}), they must satisfy certain constraints depending on the choice of ϵ, that is, depending upon whether we have discrete torsion or not. In particular, if $\epsilon = 1$, then γ_{S_1} and γ_{S_2} must commute. This follows from the fact that in this case we simply rescale the two cycles on T^2 to obtain $(T^2)'$ without the B-field, and if the matrices γ_{S_i} are non-trivial, then they act as (discrete) Wilson lines corresponding to the two cycles on T^2. However, if $\epsilon = -1$, that is, if we have non-zero B-field ($B_{12} = 1/2$), then the string consistency (in particular, the closed to open string coupling consistency) requires that γ_{S_1} and γ_{S_2} must *anticommute* [8,24,23]. In this case we can still view the action of the orbifold group on the Chan-Paton Charges as having (discrete) Wilson lines, but now they are *no* longer commuting [8,23]. Thus, to summarize, the Chan-Paton matrices must satisfy

$$\gamma_{S_1}\gamma_{S_2} = \epsilon\gamma_{S_2}\gamma_{S_1} \ . \tag{6}$$

In the case without discrete torsion ($\epsilon = 1$) we thus have gauge bundles with vector structure. In the case with discrete torsion ($\epsilon = -1$) we have gauge bundles *without* vector structure (and the corresponding generalized second Stieffel-Whitney class is non-vanishing) [24].

Note that in the cases without discrete torsion the matrices γ_{S_i} can essentially be arbitrary as long as they commute and form a (projective) representation of $\mathbf{Z}_2 \otimes \mathbf{Z}_2$. In particular, their traces are not fixed - there are no tadpoles associated with the twists S_i as they are freely acting. Thus, the action of the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold on the Chan-Paton charges can be trivial, that is, $\gamma_{S_i} = I_n$, which corresponds to having trivial Wilson lines. If we do not include an orientifold plane, then the gauge symmetry is $U(n)$ with this choice of Wilson lines, while if Wilson lines are non-trivial, then the gauge group G is a subgroup of $U(n)$ with rank $r(G) = n$. In the case of D9-branes the oriented open string theory suffers from massless tadpoles, which can be canceled if we introduce the O9$^-$-plane (with the SO type of orientifold projection on the Chan-Paton charges), and choose the number n of D9-branes to be $n = 32$. Then in the case of trivial Wilson lines we have the usual $SO(32)$ gauge group, while non-trivial Wilson lines break $SO(32)$ to its subgroup G with rank $r(G) = 16$.

However, in the case with discrete torsion the situation is qualitatively different. In particular, all three twisted Chan-Paton matrices γ_{S_a}, $a = 1, 2, 3$, must be traceless. This follows from the fact that for these matrices to form a (projective) representation of $\mathbf{Z}_2 \otimes \mathbf{Z}_2$, they must satisfy

$$\gamma_{S_a}^2 = \eta_{aa} I_n \ , \tag{7}$$

$$\gamma_{S_a}\gamma_{S_b} = \eta_{ab}\gamma_{S_c} \ , \quad a \neq b \neq c \ , \tag{8}$$

where not all the nine structure constants η_{ab} are independent but satisfy the following relations:

$$\eta_{33} = -\eta_{11}\eta_{22} \ , \tag{9}$$

$$\eta_{21} = -\eta_{12} \ , \tag{10}$$

$$\eta_{13} = -\eta_{31} = \eta_{11}\eta_{12} \ , \tag{11}$$

$$\eta_{23} = -\eta_{32} = -\eta_{22}\eta_{12} \ . \tag{12}$$

Here $\eta_{11}, \eta_{22}, \eta_{12}$ *a priori* independently take values ± 1. It then follows that $\mathrm{Tr}(\gamma_a) \equiv 0$. Moreover, even if we consider the oriented open string theory, the number of D9-branes must be even: $n = 2N$. In fact, we can now see the rank reduction phenomenon mentioned in the previous section. Thus, the rank of the unbroken gauge group is no longer n but twice as small, that is, N. This follows from the fact that the Wilson lines corresponding to the two cycles of $(T^2)'$ do not commute, which can be seen from the fact that γ_{S_1} and γ_{S_2} *anticommute*. If we consider an unoriented open string theory, that is, if we introduce an orientifold 9-plane, the rank of the gauge group is then no longer $n/2$ (as is the case without discrete torsion), but rather $N/2$ (in the presence of an orientifold plane N must also be even). Intuitively it should be clear that to cancel all the tadpoles we must introduce the $\mathrm{O9}^-$-plane (and *not* the $\mathrm{O9}^+$-plane), and choose the number of D9-branes to be $n = 32$. However, we would like to derive this result rigorously as understanding this point will be important for the subsequent discussions.

To do this, let us start with the Klein bottle amplitude $\mathcal{K}$. It is obtained from the torus amplitude by inserting the orientifold projection Ω into the trace over the Hilbert space of closed string states. This implies that the only states contributing into the Klein bottle amplitude are *left-right symmetric* closed string states. The oscillator modes are not going to be important in the following as their contributions are the same with or without the B-flux, so let us focus on the left- and right-momentum contributions. Note, in particular, that the momentum numbers m_i are invariant under the action of Ω, while the winding numbers n^i change sign under the action of Ω. This implies that only the states with zero winding (but arbitrary momentum) numbers contribute to the Klein bottle amplitude. Such states are the same regardless of the B-flux, which can be readily seen from (4). Thus, the Klein bottle amplitude is independent of the B-field. We will write the Klein bottle amplitude in the language of the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold of T^2 with discrete torsion (here we do not display the oscillator contributions):

$$\mathcal{K} = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\sum_{\mathbf{m}} q^{\frac{1}{2}P^2_{\mathbf{m}}}\left[1 + (-1)^{m_1} + (-1)^{m_2} + (-1)^{m_1+m_2}\right] , \tag{13}$$

where $P_{\mathbf{m}} = \tilde{e}^i m_i$, $\mathbf{m} = (m_1, m_2)$, and $m_i \in \mathbf{Z}$. Also, the first numerical prefactor of $(1/2)$ is related to the orientifold projection, while the second numerical prefactor of $(1/4)$ is related to the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold projection. As we have already mentioned, the Klein bottle amplitude (13) does not depend on whether we have discrete torsion or not, that is, $\mathcal{K}$ in (13) is the same as the Klein bottle amplitude for the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold *without* discrete torsion. Note that $\mathcal{K}$ is written in terms of the metric on T^2. We can rewrite it in terms of the metric on $(T^2)'$ as follows:

$$\mathcal{K} = \left(\frac{1}{2}\right)\sum_{\mathbf{m}'} q^{\frac{1}{2}P^2_{\mathbf{m}'}} , \tag{14}$$

where where $P_{\mathbf{m}'} = \tilde{e}'^i m'_i$, $\mathbf{m}' = (m'_1, m'_2)$, and $m'_i \in \mathbf{Z}$. In arriving at (14) we have explicitly performed the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold projections in (13) which keep only the states with $m_i \in 2\mathbf{Z}$. These states are then rewritten in terms of new momenta $m'_i \in \mathbf{Z}$ with the dual vielbeins $\tilde{e}'^i = 2\tilde{e}^i$ on $(T^2)'$.

Let us now discuss the annulus amplitude. We will write the latter in the language of D9-branes wrapped on T^2 with the subsequent $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold action. Thus, the annulus amplitude is given by:

$$\mathcal{A} = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\sum_{\mathbf{m}} q^{\frac{1}{2}P^2_{\mathbf{m}}}\Big[(\mathrm{Tr}(\gamma_I))^2 + (-1)^{m_1}(\mathrm{Tr}(\gamma_{S_1}))^2 + (-1)^{m_2}(\mathrm{Tr}(\gamma_{S_2}))^2 + (-1)^{m_1+m_2}(\mathrm{Tr}(\gamma_{S_3}))^2\Big] \ . \tag{15}$$

Here we have chosen a specific orientation on the annulus so that the traces over the Chan-Paton charges give $(\mathrm{Tr}(\gamma_{S_a}))^2$, while if we chose the opposite orientation, we would instead have $\mathrm{Tr}(\gamma_{S_a})\mathrm{Tr}(\gamma^{-1}_{S_a})$.

Note that $\mathrm{Tr}(\gamma_I) = n$, and $\mathrm{Tr}(\gamma_{S_a}) = 0$, $a = 1, 2, 3$, and we can *formally* rewrite the annulus amplitude via

$$\mathcal{A}_1 = \left(\frac{1}{2}\right)\sum_{\mathbf{m}} q^{\frac{1}{2}P^2_{\mathbf{m}}}(\mathrm{Tr}(I_N))^2 \ , \tag{16}$$

so that naively one could reinterpret this annulus amplitude as corresponding to that of $N = n/2$ D9-branes (instead of n D9-branes). This interpretation, however, would be erroneous. Indeed, one should get a hint of this from the sum over the momenta $\mathbf{m}$. These are the momenta written in terms of the metric on the original torus T^2. However, we would have to rewrite it in terms of the metric on the torus $(T^2)'$ - after all we are considering D9-branes wrapped $(T^2)'$ (plus the B-field), and *not* on T^2, the latter merely being the starting point for the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold with discrete torsion which is a way of obtaining $(T^2)'$ with the B-field. However, if we chose to reinterpret the annulus partition function via (16), where we essentially would be trying to ameliorate all the traces of the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold, we would have to write the momenta in terms of the metric on $(T^2)'$. Thus, we would ultimately arrive at the conclusion that the momenta would have to take *half*-integer (instead of usual integer) values. Indeed, $P_{\mathbf{m}}$ in (16) is given by $P_{\mathbf{m}} = \widetilde{e}^i m_i$, which can be rewritten in terms of the dual vielbeins $\widetilde{e}'^i$ on $(T^2)'$ as $P_{\hat{\mathbf{m}}} = \widetilde{e}'^i \hat{m}_i$, where $\hat{m}_i \equiv m_i/2$ (recall that $\widetilde{e}'^i = 2\widetilde{e}^i$), so that the new momenta $\hat{m}_i$ would be half-integer. This signals that such an interpretation would indeed be erroneous[5]. (In particular, note that in the Klein bottle amplitude (14) the sum is over integer momenta m'_i written in terms of the metric on $(T^2)'$, while in the annulus amplitude (16) the sum is over *half*-integer momenta $\hat{m}_i$.) Instead, the correct physical interpretation is that we have $n = 2N$ (and *not* N) D9-branes. The effect of the B-field then can be understood as in the interpretation provided by the annulus amplitude (15) via the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold construction. Thus, we see that merely

[5]This erroneous interpretation has been adopted in most of the literature on orientifolds with non-zero B-flux. Often it does not lead to inadequate description of the massless spectra of such orientifolds. For instance, this interpretation, which was essentially adopted in [21], gives the correct results in the case of toroidal orientifolds with B-flux due to their relative simplicity (these theories have 16 supercharges). In fact, this can be understood from the fact that the annulus amplitude in (16) would give the same massless spectrum as that in (15). Moreover, it would predict the same degeneracy of states at the massive KK levels as (15), but the vertex operators at the massive levels in the two interpretations would be different. As we will see in the following, in more complicated cases such as K3 or Calabi-Yau orientifolds with B-flux knowing the correct structure of vertex operators becomes relevant already at the *massless* level, for instance, in the 59 open string sector.

examining the partition functions is not sufficient to understand the geometric structure of orientifolds with B-field as the former only provide us with the multiplicities of states at various string levels, but may not carry the complete information about the structure of the vertex operators.

At any rate, let us start with the annulus amplitude (15), and proceed further. Thus, we would like to discuss the Möbius strip amplitude next. It is given by

$$\mathcal{M} = \lambda \left(\frac{1}{2}\right) \left(\frac{1}{4}\right) \sum_{\mathbf{m}} q^{\frac{1}{2}P^2_{\mathbf{m}}} \Big[\mathrm{Tr}(\gamma_\Omega^{-1}\gamma_\Omega^T) + (-1)^{m_1}\mathrm{Tr}(\gamma_{\Omega S_1}^{-1}\gamma_{\Omega S_1}^T) + \\ (-1)^{m_2}\mathrm{Tr}(\gamma_{\Omega S_2}^{-1}\gamma_{\Omega S_2}^T) + (-1)^{m_1+m_2}\mathrm{Tr}(\gamma_{\Omega S_3}^{-1}\gamma_{\Omega S_3}^T)\Big] , \qquad (17)$$

where $\lambda = \mp 1$ for the SO/Sp orientifold projection (that is, $\lambda = \pm 1$ for the O9$^\pm$-plane). We can simplify this expression by noting that

$$\mathrm{Tr}(\gamma_\Omega^{-1}\gamma_\Omega^T) = \mathrm{Tr}(\gamma_I) , \qquad (18)$$

$$\mathrm{Tr}(\gamma_{\Omega S_a}^{-1}\gamma_{\Omega S_a}^T) = \mathrm{Tr}(\gamma_{(S_a)^2}) = \eta_{aa}\mathrm{Tr}(\gamma_I) . \qquad (19)$$

In the second line we have used the fact that $\gamma_{(S_a)^2} = \gamma^2_{S_a} = \eta_{aa} I_n$. Thus, the Möbius strip amplitude is given by

$$\mathcal{M} = \lambda \left(\frac{1}{2}\right) \left(\frac{1}{4}\right) \sum_{\mathbf{m}} q^{\frac{1}{2}P^2_{\mathbf{m}}} \mathrm{Tr}(\gamma_I)\Big[1 + \eta_{11}(-1)^{m_1} + \eta_{22}(-1)^{m_2} + \eta_{33}(-1)^{m_1+m_2}\Big] , \qquad (20)$$

where $\eta_{33} = -\eta_{11}\eta_{22}$.

Note that the quantity in the square brackets in (20) is always $+2$ or -2. Thus, we can write

$$1 + \eta_{11}(-1)^{m_1} + \eta_{22}(-1)^{m_2} + \eta_{33}(-1)^{m_1+m_2} \equiv 2\rho_{\mathbf{m}}(\eta_{aa}) , \qquad (21)$$

where $\rho_{\mathbf{m}}(\eta_{aa}) = \pm 1$. In fact, for a given choice of η_{aa}, $\rho_{\mathbf{m}}$ by definition only depends on whether m_1 and m_2 are even or odd. So *formally* we can rewrite the Möbius strip amplitude as follows:

$$\mathcal{M}_1 = \lambda \left(\frac{1}{2}\right) \sum_{\mathbf{m}} q^{\frac{1}{2}P^2_{\mathbf{m}}} \mathrm{Tr}(I_N)\rho_{\mathbf{m}}(\eta_{aa}) . \qquad (22)$$

Thus, naively one could reinterpret this Möbius strip amplitude along the lines of (16) as corresponding to that of $N = n/2$ D9-branes (instead of n D9-branes), with the SO/Sp orientifold projection at integer KK levels $\hat{m}_i$ (recall that $\hat{m}_i \equiv m_i/2$) if $\lambda\rho_{\mathbf{m}}(\eta_{aa}) = \mp 1$ for $m_1, m_2 \in 2\mathbf{Z}$, while at the half-integer KK levels $\hat{m}_i$ the type of the orientifold projection is determined by the corresponding signs $\lambda\rho_{\mathbf{m}}(\eta_{aa})$ with either m_1 or m_2 or both odd. Such an interpretation, however, would be erroneous for the same reasons as in the annulus case discussed above.

Next, we turn to the tadpole cancellation conditions. To extract the tadpoles, we must rewrite the open string *loop-channel* Klein bottle $\mathcal{K}$, annulus $\mathcal{A}$ and Möbius strip $\mathcal{M}$ amplitudes in terms of the corresponding closed string *tree-channel* exchange expressions. In doing so, as usual, one must be careful with the relative normalizations between the proper times on these three surfaces. The modular transformations that map the loop-channel

expressions to the tree-channel expressions amount to Poisson resummations of the momentum sums in $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{M}$ (they also act non-trivially on the characters corresponding to the oscillator contributions). It is not difficult to see that after Poisson resummations the terms in (13), (15) and (17) containing $(-1)^{m_1}$, $(-1)^{m_2}$ and $(-1)^{m_1+m_2}$ do *not* contain terms corresponding to the massless closed string exchange, and, therefore, do not contribute to the tadpoles. This, actually, has been anticipated from the fact that the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold is freely acting. The remaining terms, which do contribute to the tadpoles, are the same (up to a universal overall factor of 1/4) as those in the Ω orientifold of Type IIB on T^2 (with zero B-field) in the presence of n D9-branes. This makes extracting the tadpoles in our case straightforward. Thus, the tadpoles are given by (here c is a universal numerical constant):

$$\mathrm{Tad}(\mathcal{K}) = c\ \mathrm{Vol}(T^2)\ 32^2\ , \tag{23}$$

$$\mathrm{Tad}(\mathcal{A}) = c\ \mathrm{Vol}(T^2)\ (\mathrm{Tr}(\gamma_I))^2\ , \tag{24}$$

$$\mathrm{Tad}(\mathcal{M}) = c\ \mathrm{Vol}(T^2)\ 64\lambda\ \mathrm{Tr}(\gamma_I)\ , \tag{25}$$

so that the total tadpole factorizes into a perfect square

$$\mathrm{Tad} = c\ \mathrm{Vol}(T^2)[32 + \lambda\ \mathrm{Tr}(\gamma_I)]^2\ . \tag{26}$$

Since $\mathrm{Tr}(\gamma_I) = n$, we conclude that the orientifold projection must be of the SO type ($\lambda = -1$), that is, we must include the O9$^-$-plane (and *not* the O9$^+$-plane), as well as $n = 32$ D9-branes.

Thus, we see that the number of D9-branes is indeed 32, and the orientifold plane is of the O9$^-$ type, which induces the SO type of projection on the D9-branes. This, however, does *not* imply that we cannot obtain, say, Sp gauge symmetry, which is expected to arise in orientifolds with B-flux [21]. In order to understand this point we must study the possible choices of the twisted Chan-Paton matrices γ_{S_i}, which is related to the question of possible gauge bundles on T^2 in the presence of the B-field.

To begin with let us note that it suffices to consider a 2×2 representation for γ_{S_i} as the full $2N \times 2N$ matrices can be obtained as the N-fold copy of the corresponding 2×2 matrices. Thus, we can write

$$\gamma_{S_a} = \gamma_a \otimes I_N\ , \tag{27}$$

where the matrices γ_a, $a = 1, 2, 3$, form the aforementioned 2×2 representation. These matrices must satisfy $\gamma_a^2 = \eta_{aa} I_2$ as well as $\mathrm{Tr}(\gamma_a) = 0$ conditions, which, in particular, imply that $\det(\gamma_a) = -\eta_{aa}$. Recall that $\eta_{33} = -\eta_{11}\eta_{22}$, so either all three γ_a matrices have determinant $+1$ (if $\eta_{11} = \eta_{22} = -1$), or two have determinant -1, while one has determinant $+1$ (for the other three choices of η_{11} and η_{22}). In the former case γ_a are $SU(2)$ matrices, and form a 2-dimensional representation of the non-Abelian dihedral D_4 subgroup of $SU(2)$. In the latter case γ_a form a 2-dimensional representation of the "double cover" of the D_4 subgroup of $SU(2)$, which we will denote by D_4'. Note that D_4' is not a subgroup of $SU(2)$ but is a subgroup of $SO(3)$. Up to equivalent representations, we can write γ_a for the above two cases as follows:

$$D_4: \quad \gamma_1 = i\sigma_3\ , \quad \gamma_2 = i\sigma_2\ , \quad \gamma_3 = i\eta_{12}\sigma_1\ , \tag{28}$$

$$D_4': \quad \gamma_1 = \sigma_3\ , \quad \gamma_2 = \sigma_1\ , \quad \gamma_3 = i\eta_{12}\sigma_2\ . \tag{29}$$

Here $\sigma_1, \sigma_2, \sigma_3$ are the usual 2×2 Pauli matrices. It is not difficult to show that the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold projection breaks the $SO(32)$ gauge group on 32 D9-branes at the O9$^-$-plane down to $Sp(16)$ in the D_4 case[6], and $SO(16)$ in the D_4' case. More precisely, the unbroken gauge symmetries are $Sp(16) \otimes \mathbf{Z}_2$ and $SO(16) \otimes \mathbf{Z}_2$, respectively. The extra discrete $\mathbf{Z}_2$ gauge symmetry will be important in the subsequent discussions, so let us elaborate on its appearance in more detail.

Let us first consider the D_4' case. Note that the first $\mathbf{Z}_2$ twist γ_{S_1} acting on the Chan-Paton charges breaks $SO(32)$ down to $SO(16) \otimes SO(16)$. The second $\mathbf{Z}_2$ twist γ_{S_2} breaks the latter down to its diagonal subgroup $SO(16)_{\mathrm{diag}}$ times the discrete $\mathbf{Z}_2$ gauge symmetry associated with the permutation of the two $SO(16)$ subgroups. Similarly, in the D_4 case γ_{S_1} breaks $SO(32)$ down to $U(16)$, and γ_{S_2} breaks the latter down to $Sp(16)$ times the discrete $\mathbf{Z}_2$ subgroup of the $U(1)$ subgroup of $U(16)$ (under which, for instance, the fundamental and antifundamental representations of $SU(16)$ are charged).

Before we end this subsection, a few comments are in order. First, above we have described how to obtain the $SO(16)$ and $Sp(16)$ gauge symmetries via the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold construction. Since these theories have 16 supercharges, it is clear that it should be possible to construct other points in the moduli space, whose generic points have $U(1)^8$ gauge symmetry, but at other points one should be able to obtain enhanced unitary gauge subgroups with rank 8, in particular, $U(8)$. Moreover, one expects that the moduli space of gauge bundles on T^2 without vector structure should be connected, so that it should be possible to continuously interpolate between the $SO(16)$ and $Sp(16)$ points. A detailed discussion of gauge bundles on T^2 without vector structure can be found in [24]. Here we will briefly review some of the basic relevant facts. Thus, the matrices γ_{S_i} given above correspond to points in the moduli space which can be described via the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold. However, the matrices γ_{S_i} can be more generally viewed as describing the gauge bundle on T^2 without vector structure if we think about them as anticommuting Wilson lines. Thus, we can relax all the constraints we have imposed on γ_{S_i} except for the non-commutative property: $\gamma_{S_1}\gamma_{S_2} = -\gamma_{S_2}\gamma_{S_1}$. The most general solution to this constraint can be (up to equivalent representations) written as

$$\gamma_{S_i} = \gamma_i \otimes \Gamma_i \ , \tag{30}$$

where the unitary $N \times N$ matrices Γ_i commute. In particular, note that there is no longer a restriction on Γ_i^2. Thus, starting from the solution corresponding to D_4 we can smoothly interpolate to the solution corresponding to D_4' with the intermediate points corresponding to $U(N/2)$ or its subgroups of rank $N/2$.

Another point we would like to mention is generalization to higher tori with the rank of B_{ij} $b > 2$. It is clear that, say, in the case of T^4 we can represent non-zero B-field essentially along the same lines as we have done so far for T^2. Instead of being most general here, for illustrative purposes let us consider the case of $T^4 = T^2 \otimes T^2$ (generalizations should be clear). We can introduce two Wilson lines corresponding to the two cycles on the first T^2, call the corresponding Chan-Paton matrices $\gamma_{S_1}, \gamma_{S_2}$, and two other Wilson lines corresponding

[6]In our notations $Sp(2r)$ has rank r.

to the two cycles on the second T^2, call the corresponding Chan-Paton matrices $\gamma_{T_1}, \gamma_{T_2}$. If we take γ_{S_1} and γ_{S_2} anticommuting, as well as γ_{T_1} and γ_{T_2} anticommuting, but γ_{S_i} and γ_{T_j} commuting, this corresponds to rank $b = 4$ half-integer B-field on T^4. Such Chan-Paton matrices can be easily constructed. Thus, let γ_a be a set of three anticommuting 2×2 matrices, and let β_a be another set of three anticommuting 2×2 matrices, each set satisfying the constraints discussed above in the T^2 case. Then we can choose the Chan-Paton matrices corresponding to the four Wilson lines on T^4 as follows ($N' \equiv n/4$):

$$\gamma_{S_a} = \gamma_a \otimes I_2 \otimes I_{N'} \ , \tag{31}$$

$$\gamma_{T_a} = I_2 \otimes \beta_a \otimes I_{N'} \ . \tag{32}$$

Note that the type of the unbroken gauge group can be determined as follows. If γ_a and β_a both are of the D_4' or D_4 type, then the unbroken gauge group is $SO(N')$. If one of them is of the D_4' type while the other one is of the D_4 type, then the unbroken gauge group is $Sp(N')$. (More precisely, the unbroken gauge symmetry includes the corresponding $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ subgroup.) Other points in the moduli space interpolating between these special configurations can be obtained in complete parallel with the T^2 case. Also, generalizations to higher tori, in particular, T^6 should be evident from the above discussions.

Finally, note that, say, in the case of T^2 having two non-commuting Wilson lines implies that some of the components of the non-Abelian gauge field strength F_{12} (in the language of the original $SO(32)$ gauge symmetry) in the compact directions are non-zero. (In the four dimensional language this implies that some of the D-term components are non-zero.) This is precisely the statement that the corresponding generalized second Stieffel-Whitney class of the gauge bundle in non-vanishing. Note that this is perfectly consistent with the low energy supersymmetry as the moduli corresponding to these directions are absent in the low energy spectrum of the $SO(16)$ or $Sp(16)$ gauge theory with 16 supercharges. In particular, the moduli space of gauge bundles of the $SO(32)$ theory on T^2 with vector structure is 2×16 dimensional, whereas that of gauge bundles of the $SO(16)/Sp(16)$ theory on T^2, which lacks vector structure, is 2×8 dimensional. In fact, these two components of the moduli space of gauge bundles on T^2 are disconnected. Under Type I-heterotic duality the component with vector structure maps to the corresponding part of the Narain moduli space, while, as was originally pointed out in [8], the component without vector structure maps to the corresponding part of the moduli space of CHL strings [28] (with rank 8 gauge symmetry) in 8 dimensions.

B. D-branes and O-planes Transverse to Tori with B-flux

In this subsection we will discuss toroidal orientifolds with D-branes and O-planes transverse to tori with non-zero B-field. From [24] we expect that in these cases we will have both O^--planes and O^+-planes. We will see that this is indeed the case using the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold of [8,23] described in the previous subsection. However, before we discuss orientifolds with non-zero B-flux, we will take a detour into a discussion of possible types

of O$^{\pm}$-planes as there are two different types of O^{+}-planes as well as O^{-}-planes[7].

To begin with, let us start with Type IIB in ten dimensions. *A priori* we can orientifold Type IIB by two different orientifold actions, which we will refer to as $\Omega_\pm$. One of these actions, namely, Ω_- (which we have been denoting by Ω in the previous discussions) is the usual orientifold projection which leads to the Type I theory in ten dimensions with 16 supercharges and $SO(32)$ gauge group. Note that the action of Ω_- on the Chan-Paton charges of D9-branes is *antisymmetric* in both Neveu-Schwarz and Ramond open string sectors. The corresponding orientifold plane has R-R charge -32, which requires introduction of 32 D9-branes to cancel the R-R charge as each D9-brane has R-R charge $+1$. The R-R charge cancellation implies that the corresponding R-R tadpoles are also canceled. The fact that the NS-NS tadpoles also cancel then follows from supersymmetry. However, in cases without supersymmetry NS-NS and R-R tadpole cancellations are *a priori* independent. To keep track of both NS-NS and R-R tadpoles it is convenient to introduce the notion of the "NS-NS charge" for D-branes and O-planes as a *book-keeping* device (albeit there is no real conserved charge associated with NS-NS tadpoles). Thus, let us assign NS-NS charge $+1$ to a single D9-brane. Then the NS-NS charge of the orientifold plane corresponding to the action of Ω_- is -32. In this language the orientifold plane corresponding to the action of Ω_+ has R-R charge $+32$ and NS-NS charge $+32$. The action of Ω_+ on the Chan-Paton charges of D9-branes is now *symmetric* in both NS and R open string sectors. Note that the gauge group on M D9-branes with the O-plane corresponding to the action of Ω_+ is $Sp(M)$. In fact, the corresponding gauge theory is supersymmetric. However, in ten dimensions such a theory would be anomalous as we cannot cancel the R-R charge: both the O-plane and D9-branes have positive R-R charges. Here we can ask if we could cancel the R-R charges by introducing anti-D9-branes, which we will denote via D$\overline{9}$. Each D$\overline{9}$-brane has R-R charge -1 and NS-NS charge $+1$. So if we introduce 32 D$\overline{9}$-branes together with the aforementioned O-plane, we can cancel the R-R tadpoles, so that the resulting theory would be anomaly free. However, this theory would be non-supersymmetric, and NS-NS tadpoles would not be canceled.

Let us systematize the above discussion. Let us denote the NS-NS and R-R charges of a given object by $Q_{\rm NS}$ and $Q_{\rm R}$, respectively. Then D9-branes and D$\overline{9}$-branes have the following $(Q_{\rm NS}, Q_{\rm R})$ charges:

$$\mathrm{D9-brane:}\quad (+1,+1)\ , \tag{33}$$
$$\mathrm{D\overline{9}-brane:}\quad (+1,-1)\ . \tag{34}$$

We can introduce *four* different types of O9-planes according to their NS-NS and R-R charges:

$$\mathrm{O9}^{--}:\quad (-32,-32)\qquad \Omega_-\ , \tag{35}$$
$$\mathrm{O9}^{++}:\quad (+32,+32)\qquad \Omega_+\ , \tag{36}$$
$$\mathrm{O9}^{+-}:\quad (+32,-32)\qquad \Omega_+(-1)^{F_L+F_R}\ , \tag{37}$$
$$\mathrm{O9}^{-+}:\quad (-32,+32)\qquad \Omega_-(-1)^{F_L+F_R}\ , \tag{38}$$

[7]I would like to thank Amihay Hanany for a valuable discussion on this point.

where at the end of each line we have indicated the type of the Type IIB orientifold which produces each of these objects (F_L and F_R are the usual left- and right-moving space-time fermion numbers). We will discuss the above four orientifolds more explicitly in a moment. However, before we do this, let us discuss the action of the orientifold projections on D9- and D$\overline{9}$-branes induced by the above orientifold planes. In the following "S" refers to symmetrization on Chan-Paton charges, while "A" refers to antisymmetrization. The first entry corresponds to the NS (that is, space-time bosonic) open string sector on the corresponding D9/D$\overline{9}$-branes, while the second entry corresponds to the R (that is, space-time fermionic) sector. The orientifold projections of the above O9-planes on the D9-branes are given by:

$$\mathrm{O9}^{--}: \quad (A, A)\ , \qquad (-32+M, -32+M)\ , \tag{39}$$
$$\mathrm{O9}^{++}: \quad (S, S)\ , \qquad (32+M, 32+M)\ , \tag{40}$$
$$\mathrm{O9}^{+-}: \quad (S, A)\ , \qquad (32+M, -32+M)\ , \tag{41}$$
$$\mathrm{O9}^{-+}: \quad (A, S)\ , \qquad (-32+M, 32+M)\ , \tag{42}$$

where at the end of each line we have indicated the total NS-NS and R-R charges of the background with the corresponding O9-plane and M D9-branes. Let us discuss these four cases in more detail.

• $\mathrm{O9}^{--}$-plane plus M D9-branes. The theory is supersymmetric, the gauge group is $SO(M)$, and the fermions on D9-branes are in the antisymmetric representation. The R-R tadpoles cancel for $M = 32$, that is, we have an anomaly free theory in this case. Note that NS-NS tadpoles also cancel. This is the familiar Type I theory with $SO(32)$ gauge symmetry.

• $\mathrm{O9}^{++}$-plane plus M D9-branes. The theory is supersymmetric, the gauge group is $Sp(M)$, and the fermions on D9-branes are in the symmetric representation. The R-R (and NS-NS) tadpoles, however, cannot be canceled, so that the theory is anomalous.

• $\mathrm{O9}^{+-}$-plane plus M D9-branes. The closed string sector is supersymmetric at the tree level. The open string sector is non-supersymmetric. The gauge group is $Sp(M)$, and the fermions on the D9-branes are in the antisymmetric representation. The R-R tadpoles cancel for $M = 32$, that is, we have an anomaly free theory in this case. This is the non-supersymmetric $Sp(32)$ string theory recently proposed in [29]. Note, however, that NS-NS tadpoles do *not* cancel in this theory, so that the flat Minkowski metric with constant dilaton (more precisely, the corresponding $\mathcal{N} = 1$ supergravity) does not appear to be the correct background for this theory. At present it is unclear whether there is a consistent background for this theory which could be reached via the Fischler-Susskind mechanism.

• $\mathrm{O9}^{-+}$-plane plus M D9-branes. The closed string spectrum is supersymmetric at the tree level. The open string sector is non-supersymmetric. The gauge group is $SO(M)$, and the fermions on D9-branes are in the symmetric representation. The NS-NS tadpoles cancel for $M = 32$, but the R-R tadpoles cannot be canceled, so that the theory is anomalous.

Next, let us discuss the orientifold projections induced by the above O9-planes on D$\overline{9}$-branes:

$$\mathrm{O9}^{--}: \quad (A, S)\ , \qquad (-32+M, -32-M)\ , \tag{43}$$
$$\mathrm{O9}^{++}: \quad (S, A)\ , \qquad (32+M, 32-M)\ , \tag{44}$$
$$\mathrm{O9}^{+-}: \quad (S, S)\ , \qquad (32+M, -32-M)\ , \tag{45}$$
$$\mathrm{O9}^{-+}: \quad (A, A)\ , \qquad (-32+M, 32-M)\ , \tag{46}$$

where, as before, at the end of each line we have indicated the corresponding total NS-NS and R-R tadpoles for the system of the corresponding O9-plane and M D$\overline{9}$-branes. It is not difficult to see that the system of the O$9^{\alpha,\beta}$-plane plus M D$\overline{9}$-branes gives the same theory as the system of the O$9^{\alpha,-\beta}$-plane plus M D9-branes, where $\alpha, \beta = \pm$.

Here we would like to make one remark. In the above language we can describe the Type O open plus closed string theory, which is the Ω_- orientifold of Type 0B string theory, as follows. Type 0B can be viewed as the $(-1)^{F_L+F_R}$ orbifold of Type IIB. This implies that Type O can be viewed as a Type IIB orientifold, where the orientifold group is $\mathcal{O} = \{1, (-1)^{F_L+F_R}, \Omega_-, \Omega_-(-1)^{F_L+F_R}\}$. Thus, we have two orientifold planes in this theory, namely, the O9^{--}-plane and the O9^{-+}-plane, which together have the following NS-NS and R-R charges: $(Q_{\rm NS}, Q_R) = (-64, 0)$. These can be canceled by introducing 32 D9-branes together with 32 D$\overline{9}$-branes[8]. Note that in this language, for instance, it is evident why there are no fermionic states in the open string spectrum: the O9^{--} orientifold projection on D9-brane fermions is antisymmetric, while the O9^{-+} orientifold projection on the same fermions is symmetric, so that fermions are completely projected out. The same conclusion holds for the D$\overline{9}$-brane fermions.

The reason why we discussed four different kinds of O-planes is that in our discussion of orientifolds with B-flux we will encounter O-planes with the Sp type of orientifold projection. However, as we have already pointed out, one must distinguish two possible O-planes of this type as well as two O-planes with the SO type of orientifold projection. In the following we will only encounter O-planes of O^{--} and O^{++} type[9]. We will refer to them as O$^-$ and O$^+$, respectively. Also, just as in the previous sections, we will refer to Ω_- as Ω. Note that the NS-NS and R-R charges for each of these O-planes are equal. The same holds for D-branes (we will not need to introduce anti-D-branes in the following), so in the following it will suffice to just consider R-R charges. This is simply a manifestation of the fact that all the theories we will consider in the following are supersymmetric.

Next, we would like to return to orientifolds with B-flux, and consider cases where O-planes and the corresponding D-branes are transverse to tori with non-zero B-flux. To begin with, let us start with the simplest case. Thus, consider the $\Omega R(-1)^{F_L}$ orientifold of Type IIB on $\mathbf{R}^{1,7} \otimes T^2$, where R acts as $RX_{1,2} = -X_{1,2}$ on the compact coordinates, and the B-flux on T^2 is zero. To consider the case with non-zero B-flux, let us use the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold construction discussed in the previous subsection. In fact, we will discuss both cases with and without discrete torsion, whose comparison will be helpful in understanding the types of orientifold planes arising in the former case.

Thus, let us consider the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold of T^2 whose generators S_1 and S_2 act as shifts $S_i X_i = X_i + e_i/2$. Note that there are four fixed points of R on T^2 located at

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[8] In Type 0B, unlike Type IIB, there are two different types of D-branes and anti-D-branes, so other solutions to the tadpole cancellation conditions also exist [30].

[9] In the following we will discuss Op-planes with $p < 9$ for which the above discussion of the possible types of O-planes can be straightforwardly generalized. In particular, we can start from the aforementioned four types of O9-planes and use the standard T-duality arguments to arrive at the corresponding lower dimensional Op-planes.

$(X_1, X_2) = (0,0), (e_1/2, 0), (0, e_2/2), (e_1/2, e_2/2)$. However, these four fixed points are identified by the combined action of the S_1 and S_2 shifts, so that we expect one orientifold 7-plane at $(X_1, X_2) = (0,0)$. On the other hand, we have additional fixed points due to the elements RS_a, $a = 1,2,3$ (recall that $S_3 \equiv S_1 S_2$). In particular, for a given element RS_a we have four fixed points, which are identified by the shifts S_1 and S_2. So we have one independent fixed point corresponding to each element RS_a: $(X_1, X_2) = (e_1/4, 0)$ for RS_1, $(X_1, X_2) = (0, e_2/4)$ for RS_2, and $(X_1, X_2) = (e_1/4, e_2/4)$ for RS_3. At each of these three fixed points we expect one orientifold 7-plane.

Next, we would like to understand what types of O7-planes we have in this background. In the case without discrete torsion we have an $\mathrm{O7}^-$-plane at each of the aforementioned four fixed points. In the case with discrete torsion, however, one of the O7-planes is of the $\mathrm{O7}^+$ type, while the other three are of the $\mathrm{O7}^-$ type. To show this, let us consider the Klein bottle in each case. Thus, we have four sectors of the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orientifold labeled by $(\alpha^1, \alpha^2) = (0,0), (1,0), (0,1), (1,1)$, and the left- and right-moving closed string momenta are given in these sectors by (2), where the windings n^i are arbitrary integers, whereas the momenta m_i are integers subject to the orbifold projections which can be read off (3). Let us denote the Klein bottle contribution coming from a given sector (α^1, α^2) via $\mathcal{K}(\alpha^1, \alpha^2)$. Let us denote the contribution coming from the tree-level closed string exchange between the O7-plane located at the fixed point labeled by (β^1, β^2) and the O7-plane located at the fixed point labeled by (γ^1, γ^2) via $\widetilde{\mathcal{K}}(\beta^1, \beta^2; \gamma^1, \gamma^2)$. Here the fixed point labeled by (β^1, β^2) is given by $(X_1, X_2) = (\beta^1 e_1/4, \beta^2 e_2/4)$, $\beta^i = 0, 1$. Then we have the following relation:

$$\mathcal{K}(\alpha^1, \alpha^2) = \sum_{\beta^1, \beta^2 = 0,1} \widetilde{\mathcal{K}}(\beta^1, \beta^2; \beta^1 + \alpha^1 \ (\mathrm{mod}\ 2), \beta^2 + \alpha^2 \ (\mathrm{mod}\ 2)) \ . \qquad (47)$$

Note that for $\epsilon = 1$ in the untwisted as well as twisted sectors the momenta m_i are even integers. This implies that both the untwisted as well as twisted sector contributions $\mathcal{K}(\alpha^1, \alpha^2)$ are non-vanishing - the $\Omega R(-1)^{F_L}$ orientifold projection keeps states with zero momenta and arbitrary windings. This implies that all four orientifold planes must have the same R-R charge. This can be seen explicitly by extracting the massless R-R tadpoles in each of these sectors, which are actually identical. (To see this one can perform the Poisson resummation of the corresponding winding sums.) However, for $\epsilon = -1$ the situation is quite different. Thus, in the untwisted sector $(\alpha^1, \alpha^2) = (0,0)$ the momenta m_i are even, so that the corresponding Klein bottle contribution $\mathcal{K}(0,0)$ is non-vanishing. On the other hand, in the twisted sectors $(\alpha^1, \alpha^2) = (1,0), (0,1), (1,1)$ we have the momenta $(m_1 \in 2\mathbf{Z}, m_2 \in 2\mathbf{Z}+1)$, $(m_1 \in 2\mathbf{Z}+1, m_2 \in 2\mathbf{Z})$ and $(m_1 \in 2\mathbf{Z}+1, m_2 \in 2\mathbf{Z}+1)$, respectively. This implies that the corresponding Klein bottle contributions $\mathcal{K}(\alpha^1, \alpha^2)$ vanish, and so do the corresponding tadpoles. This is enough to deduce the R-R charges of the orientifold planes located at the four fixed points. Let $Q_R(\beta^1, \beta^2)$ be the R-R charge of the O7-plane located at the fixed point labeled by (β^1, β^2). Then we have the following conditions:

$$\sum_{\beta^1, \beta^2 = 0,1} Q_R^2(\beta^1, \beta^2) = 4 \times 8^2 \ , \qquad (48)$$

$$\sum_{\beta^1, \beta^2 = 0,1} Q_R(\beta^1, \beta^2) Q_R(\beta^1 + \alpha^1 \ (\mathrm{mod}\ 2), \beta^2 + \alpha^2 \ (\mathrm{mod}\ 2)) = 0 \ , \quad (\alpha^1, \alpha^2) \neq (0,0) \ . \qquad (49)$$

The first line follows from the fact that the untwisted sector contribution $\mathcal{K}(0,0)$ is the same as in the case without discrete torsion where all four orientifold 7-planes are of the

same type, and carry the R-R charge -8 or $+8$ depending on the choice of the orientifold projection. It is not difficult to see that the solutions to the above conditions correspond to having one O7-plane with R-R charge ± 8 and the other three O7-planes with R-R charge ∓ 8 (where the signs are correlated). The solution that we are interested in is the one where we have one $O7^+$-plane and three $O7^-$-planes as in this case we can cancel the total R-R charge of the O7-planes, which is -16, by introducing 16 D7-branes[10]. This is precisely the result we wished to show.

Note that unlike the case of, say, D9-branes wrapped on a two-torus with B-field, the "rank reduction" here is *not* due to non-commuting Wilson lines but rather the fact that we have different types of O7-planes whose total R-R charge adds up to -16 (rather than -32). In contrast, in the case of D9-branes wrapped on a two-torus with B-flux the O9-plane is of the $O9^-$ type, and the R-R charge cancellation requires 32 D9-branes. The rank reduction in this case is due to the non-commuting Wilson lines. The difference between these two cases is quite substantial. Thus, in the case of D9-branes the gauge symmetry is $SO(16) \otimes \mathbf{Z}_2$ or $Sp(16) \otimes \mathbf{Z}_2$. In the case of D7-branes the gauge symmetry is $SO(16)$ if all D7-branes are placed at one of the $O7^-$-planes, and it is $Sp(16)$ if all D7-planes are placed at the $O7^+$-plane[11]. Thus, the gauge symmetry in the case of D7-branes does not contain a $\mathbf{Z}_2$ subgroup present in the case of D9-branes.

Before we end this subsection, let us mention the generalization to higher tori. For instance, consider the case of T^4. For illustrative purposes let us actually concentrate on $T^4 = T^2 \otimes T^2$. We can obtain the background with rank $b = 4$ B-flux via separately orbifolding the two T^2's by the respective $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ actions with discrete torsion. Then, if we consider the ΩR orientifold of Type IIB in this background, where R inverts all four coordinates on T^4, we have 10 $O5^-$-planes and 6 $O5^+$-planes. Similarly, in the case of the $\Omega R(-1)^{F_L}$ orientifold of Type IIB on T^6 with $b = 6$, where R inverts all six coordinates on T^6, we have 36 $O3^-$-planes and 28 $O3^+$-planes. More generally, consider Op-planes transverse to $\mathbf{R}^{9-p-d} \otimes T^d$ with B-flux of rank b on T^d ($b \leq d$). We have total of $n_f = 2^d$ fixed points. Let the numbers of O$p^{\pm}$-planes be $n_{f\pm}$. Then $n_f = n_{f+} + n_{f-}$, and $n_{f-} - n_{f+} = 2^{d-b/2}$. Thus, we have

$$n_{f\pm} = \frac{1}{2}\left(2^d \mp 2^{d-b/2}\right) \ . \qquad (50)$$

It is also straightforward to consider the cases where D-branes and O-planes are wrapped on tori with non-zero B-field, and, at the same time, the tori transverse to these objects also have non-zero B-field by combining the results of this and the previous subsections.

[10]For the other solution where we have one $O7^-$-plane and three $O7^+$-plane the total R-R charge is $+16$, so it cannot be canceled by introducing only D7-branes.

[11]In these configurations tadpoles are not canceled locally, so one expects a varying dilaton background. Note that this is not the case in analogous configurations involving Op-planes and Dp-branes with $p < 7$ as they are non-dilatonic.

C. Mixed Cases

In this subsection we would like to discuss the cases where we have a Dp-Dp' system (with $p - p' = 0$ (mod 4) so that we preserve some supersymmetries), where one set of branes is wrapped on a torus with non-zero B-field, while the other set of branes is transverse to this torus. The most general case is straightforward to treat, however, for illustrative purposes let us consider the case of the D5-D9 system, where D9-branes are wrapped on T^4, while D5-branes are transverse to T^4.

Thus, let us assume that we have n_5 D5-branes and n_9 D9-branes. In fact, in this subsection we will not worry about tadpole/anomaly cancellation as the point we would like to make here is purely geometric, and the former issues are irrelevant for this discussion. So we will not introduce any orientifold planes. Moreover, here we will focus on the 59 sector states, so that the conclusions we draw in this subsection are unchanged even after introduction of orientifold planes (recall, for instance, that the 59 sector states do not contribute to the Möbius strip amplitude).

To begin with, let us consider the above system with zero B-flux on T^4. Then the gauge group is $U(n_5) \otimes U(n_9)$, and the 59 open string sector hypermultiplets transform in the bifundamental representation (n_5, n_9) (we will suppress the $U(1)$ charges which are straightforward to restore). Now let us turn on a half-integer B-flux of rank $b = 2$. For the sake of simplicity let us consider the case of $T^4 = T^2 \otimes T^2$ with non-zero B-flux in the first T^2. Then, as in the previous subsections, we can view D9-branes wrapped on T^4 with B-flux in terms of turning on non-commuting Wilson lines corresponding to the two cycles of the first T^2. The 55 gauge group is unchanged as the Wilson lines are turned on in the directions transverse to D5-branes. The 99 gauge group, however, is broken down to $U(N) \otimes \mathbf{Z}_2$, where $N \equiv n_9/2$. (Recall that the first Wilson line brakes $U(n_9)$ to $U(N) \otimes U(N)$, while the second Wilson line breaks the latter to its diagonal subgroup $U(N) \otimes \mathbf{Z}_2$.) Even though the 99 gauge group is broken by the Wilson lines, the 59 sector states are not affected (we will explain this in a moment). In particular, the number of the 59 hypermultiplets is still $n_5 n_9$. More concretely, we have two hypermultiplets in (n_5, N) of $U(n_5) \otimes U(N)$. Actually, the gauge group is $U(n_5) \otimes U(N) \otimes \mathbf{Z}_2$, and the 59 hypermultiplets are given by $(n_5, N)_{+1}$ and $(n_5, N)_{-1}$, where the subscript indicates the corresponding $\mathbf{Z}_2$ charge. This is precisely the phenomenon first observed in [5] - the 59 open string states come with a non-trivial multiplicity, which depends on the rank of the B-field. This multiplicity is given by

$$\xi_{59} = 2^{b/2} \ . \tag{51}$$

Here we can understand this multiplicity completely geometrically - it corresponds to the $\mathbf{Z}_2^{\otimes(b/2)}$ discrete gauge symmetry, that is, the $2^{b/2}$ states, which have otherwise identical quantum numbers, carry all different quantum numbers under the $\mathbf{Z}_2^{\otimes(b/2)}$ discrete gauge symmetry. This answers the question of the vertex operators in the 59 sector. For instance, in the $b = 2$ case we can write down the vertex operators for the 59 states as follows. The first Wilson line breaks $U(n_9)$ to $U(N) \otimes U(N)$. The original hypermultiplet in the bifundamental (n_5, n_9) of $U(n_5) \otimes U(n_9)$ thus gives rise to the following hypermultiplets charged under $U(n_5) \otimes U(N) \otimes U(N)$: $(n_5, N, 1)$ and $(n_5, 1, N)$. Next, the second Wilson line breaks $U(N) \otimes U(N)$ down to its diagonal subgroup $U(N) \otimes \mathbf{Z}_2$. The aforementioned hypermultiplets can now be combined into two linear combinations

$$|(n_5, N)_{\pm 1}\rangle = \frac{1}{\sqrt{2}} \left(|(n_5, N, 1)\rangle \pm |(n_5, 1, N)\rangle \right) \ , \tag{52}$$

which carry definite $U(n_5) \otimes U(N) \otimes \mathbf{Z}_2$ gauge quantum numbers[12].

Now let us explain why the 59 sector states are not affected by turning on Wilson lines even though the 99 sector states are. The point is that the 59 sector states have no momentum (or winding) excitations - the 99 strings have only momenta on T^4, while the 55 strings have only windings on T^4 (this is precisely why 55 sector states are unaffected by the Wilson lines which act only on the momenta). Thus, the Wilson lines do *not* act on the 59 sector states.

Before we end this section we would like to discuss one other point. As we have seen from the previous discussions, the 99 and 55 sectors feel the presence of the B-flux in qualitatively different ways. Thus, the orientifold 9-planes are unaffected by the presence of the B-field, while the D9-brane gauge symmetry suffers rank reduction. On the other hand, the structure of orientifold 5-planes is modified in the presence of the B-flux, while D5-branes are unaffected. More generally, the above conclusions hold for O-planes and D-branes wrapped on tori with B-flux *vs.* O-planes and D-branes transverse to such tori. At first this might seem puzzling in the light of T-duality, which one might expect to map the aforementioned two setups into each other. However, as we will see in a moment, such an expectation would be erroneous, and there is no puzzle here.

To understand this, let us consider the simplest case of D9-branes wrapped on T^2 with the B-flux $B_{12} = 1/2$. Let the metric on T^2 be g_{ij}. In the following it will be convenient to work with $G_{ij} \equiv 2g_{ij}$, and introduce the following matrix (here we are closely following the discussion in [31]):

$$E_{ij} \equiv G_{ij} + B_{ij} = \begin{pmatrix} G_{11} & G_{12} + B_{12} \\ G_{12} - B_{12} & G_{22} \end{pmatrix} \ . \tag{53}$$

The T-duality group in the case of T^2 is $SO(2, 2, \mathbf{Z})$ whose elements can be described in terms of 4×4 matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ , \tag{54}$$

where $\alpha, \beta, \gamma, \delta$ are 2×2 matrices with integer entries, and satisfy the following constraints:

$$\begin{aligned} &\gamma^T \alpha + \alpha^T \gamma = 0 \ , \\ &\delta^T \beta + \beta^T \delta = 0 \ , \\ &\gamma^T \beta + \alpha^T \delta = I \ . \end{aligned}$$

Here the superscript T stands for transposition, and I denotes the 2×2 identity matrix. The above T-duality element acts on E_{ij} as follows:

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[12]Actually, the appearance of the $\mathbf{Z}_2^{\otimes(b/2)}$ discrete gauge symmetry in orientifolds with B-flux was originally pointed out in [15].

$$E \to E' = (\alpha E + \beta)(\gamma E + \delta)^{-1} \ . \tag{55}$$

In these notations the familiar S- and T-transformations are described as follows. The S-transformation corresponds to taking $\alpha = 0$, $\beta = I$, $\gamma = I$ and $\delta = 0$, and amounts to mapping a 2-torus with metric G and zero B-field to another 2-torus (with zero B-field) whose metric is given by the inverse of G. This is just the usual "$R \to 1/R$" type of T-duality transformation. On the other hand, the T-transformation corresponds to taking $\alpha = I$, $\beta = \Sigma$, $\gamma = 0$ and $\delta = I$, and amounts to unit shifts of the B-field (but does not affect the metric on T^2), where Σ is the 2×2 antisymmetric matrix with $\Sigma_{12} = 1$.

Here we can ask what happens to D9-branes wrapped on T^2 with $B_{12} = 1/2$ under the S-transformation. Note that for the S-transformation the new matrix E' is simply the inverse of E, so that we have

$$E' = G' + B' = \frac{1}{\det(G) + B_{12}^2} \begin{pmatrix} G_{22} & -G_{12} - B_{12} \\ -G_{12} + B_{12} & G_{11} \end{pmatrix} \ . \tag{56}$$

Thus, the new B-field has the non-zero component given by

$$B'_{12} = -\frac{B_{12}}{\det(G) + B_{12}^2} \ . \tag{57}$$

Since $B_{12} = 1/2$, and we must require that B'_{12} must also be half-integer (or else the orientifold in the T-dual picture would not be well defined), it follows that the S-transformation can only be performed for[13] $\det(G) = 3/4$. The point in the moduli space of Kähler structures of T^2 where $B_{12} = 1/2$ and $\det(G) = 3/4$ is one of the self-dual points. That is, the momentum and winding states at this point are indistinguishable, and, therefore, D9-branes wrapped on such a 2-torus are indistinguishable from D7-branes transverse to such a torus[14]. Thus, the S-transformation, which maps D9-branes to D7-branes, can only be performed at the self-dual point where the two types of branes are identical. This avoids the aforementioned puzzle with T-duality. Indeed, at a generic point in the Kähler structure moduli space D9-branes wrapped on T^2 with B-flux are *not* T-dual to D7-branes transverse to (another) T^2 with B-flux. As to the self-dual point in the Kähler structure moduli space, there are two *a priori* consistent setups, one of which can be continuously deformed into D9-branes wrapped on a generic T^2 with half-integer B-flux, and the other one can be continuously deformed into D7-branes transverse to a generic T^2 with half-integer B-flux. The fact that these two setups are indeed different is evident - at generic points in the respective moduli

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[13] Actually, there are additional solutions to this constraint, but all of them are equivalent to the one we discuss here by T-duality transformations. Thus, for instance, we can start from the point $\det(G) = 1/12$, $B_{12} = 1/2$, which is mapped by the S-transformation to the point $\det(G') = 3/4$, $B'_{12} = -3/2$. However, the point $\det(G) = 1/12$, $B_{12} = 1/2$ is equivalent to the point $\det(G) = 3/4$, $B_{12} = 1/2$ via the T-duality transformation STS. I would like to thank Ofer Aharony for bringing this point to my attention.

[14] As we will explain in a moment, however, because of half-integer B-field the precise map between these objects is *different* from that at the self-dual point without the B-field.

spaces the number of D9-branes is *twice* the number of D7-branes for a given rank of the gauge group.

Even though the difference between the aforementioned two setups is evident, we would like to understand the origin of these two choices[15]. To do this, let us consider D9-branes wrapping T^2 with $B_{12} = 1/2$. Following [21] we can assume that the closed strings that couple to D-branes satisfy the "no momentum flow" condition in the directions of T^2, that is, these states have the left- and right-moving momenta (note that $p_i^{L,R} \equiv 2e_i \cdot P_{L,R}$)

$$p_i^L = m_i + E_{ji} n^j \ , \tag{58}$$
$$p_i^R = m_i - E_{ij} n^j \ , \tag{59}$$

which satisfy

$$p_i^L = -p_i^R \ . \tag{60}$$

This constraint implies that the closed strings coupled to D9-branes have the momenta and windings such that

$$m_i - B_{ij} n^j = 0 \ , \tag{61}$$

from which it follows that the windings n^i must be *even*[16]. This, in particular, leads to the rank reduction for the 99 Chan-Paton gauge group [21], which we have understood in terms of non-commuting Wilson lines in the previous subsections. Note that in this case we also have a consistent coupling between the D9-branes and the O9-plane. Indeed, the loop-channel Klein bottle amplitude receives contributions from the closed string states with the left- and right-moving momenta satisfying $p_i^L = p_i^R$. Thus, in the loop channel we have arbitrary momenta m_i and zero windings n^i. After the modular transformation $t \to 1/t$, which maps the loop-channel Klein bottle amplitude to the tree-channel Klein bottle amplitude, we obtain a sum over arbitrary integer windings with the metric g_{ij}. That is, the closed string states that couple to the O9-plane have the left- and right-moving

[15] Parts of our discussion here have appeared in a footnote in [32].

[16] This is in accord with (16). More precisely, (16) gives the loop-channel annulus amplitude. Upon the modular transformation $t \to 1/t$, where t is the proper time on the cylinder, which involves the appropriate Poisson resummation of the momentum sum in (16), we arrive at the tree-channel annulus amplitude corresponding to the closed string exchanges between D-branes. The latter amplitude is in agreement with the aforementioned conclusion that the closed string states that couple to D-branes have even windings. Note that, as we explained in detail in subsection A, strictly speaking the interpretation corresponding to (16), which arises in the approach of [21], is somewhat imprecise. However, it suffices for our purposes here. The above analysis can be repeated in the language of the freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold discussed in the previous subsections, which gives the precise description of D-branes wrapped on T^2 with half-integer B-field. Here we have chosen the approach of [21] for illustrative purposes, as it suffices to explain the point we are trying to make here.

momenta $p^L_i = -p^R_i = G_{ij}n^j$, so that $P_L = -P_R = e_i n^i$, and $P^2_L = P^2_R = g_{ij}n^in^j = \frac{1}{2}G_{ij}n^in^j$. (Note that these states are the same as in the case without the B-field.) Thus, the closed string states that couple to D9-branes wrapped on T^2 with half-integer B-field are a subset of the closed string states that couple to the O9-plane, so that D9-branes consistently couple to the O9-planes, in particular, the Möbius strip amplitude is consistent[17].

However, *a priori* there is another choice we can make for the constraint on the left- and right-moving momenta of the closed string states that couple to D-branes. This second choice has been recently discussed in [33], and is given by[18]

$$p^L_i = -\mathcal{R}_i{}^j p^R_j \ , \tag{62}$$

where $\mathcal{R} \equiv E^T E^{-1}$. Note that (62) reduces to (60) for $B_{ij} = 0$, but is different otherwise. It is not difficult to show that the solution of (62) is as follows. The closed string states coupled to D-branes have zero momenta m_i and *arbitrary* windings n^i (and in this case we do *not* expect rank reduction). This implies that these states have the following left- and right-moving momenta:

$$p^L_i = E_{ji}n^j \ , \quad p^R_i = -E_{ij}n^j \ . \tag{63}$$

Note that these states have the expected property

$$P^2_L = P^2_R = \frac{1}{2}\mathcal{G}_{ij}n^in^j \ , \tag{64}$$

where

$$\mathcal{G} \equiv G - BG^{-1}B \ . \tag{65}$$

Note, however, that at generic points (for half-integer B-flux) these states are quite different from the states that couple to the O9-plane for which from the above discussion we have $P^2_L = P^2_R = \frac{1}{2}G_{ij}n^in^j$. In fact, these two sets of closed string states coincide only for zero B-flux. Thus, D9-branes defined via (62) cannot be consistently coupled to the O9-plane for half-integer B-field.

There is, however, a setup where we can consistently couple such D9-branes to orientifold planes, except that these are not O9- but O7-planes. Thus, consider D7-branes transverse to T^2 with half-integer B-flux. Recall that in the case of D9-branes wrapped on such a T^2 we have imposed the "no momentum flow" condition (60). In the case of D7-branes transverse to such a T^2 the analogous condition is that of "no winding flow" [5]:

$$p^L_i = p^R_i \ . \tag{66}$$

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[17] Once again, as we explained in subsection A, a more precise description of this coupling is given in terms of the freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold, but the above description is adequate for our purposes here.

[18] There is a misprint in [33], which amounts to a missing minus sign in (62).

This constraint implies that n^i are zero and m_i are arbitrary. Thus, the left- and right-moving momenta in this case are given by $p_i^L = p_i^R = m_i$, and

$$P_L^2 = P_R^2 = \frac{1}{2} G^{ij} m_i m_j \ , \tag{67}$$

where G^{ij} is the inverse of G_{ij}. Let us compare (64) and (67). They are *identical* at the self-dual point in the moduli space of Kähler structures where $\det(G) = 3/4$ and $B_{12} = 1/2$. Indeed, note that at a generic point in this moduli space we have

$$\mathcal{G}_{ij} = \left(1 + \frac{B_{12}^2}{\det(G)}\right) G_{ij} \ . \tag{68}$$

At the self-dual point we have

$$\mathcal{G}_{ij} = \frac{4}{3} G_{ij} = \frac{4}{3} \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} . \tag{69}$$

On the other hand, for the inverse metric G^{ij} at the self-dual point we have

$$G^{ij} = \frac{1}{\det(G)} \begin{pmatrix} G_{22} & -G_{12} \\ -G_{12} & G_{11} \end{pmatrix} = \frac{4}{3} \begin{pmatrix} G_{22} & -G_{12} \\ -G_{12} & G_{11} \end{pmatrix} \ . \tag{70}$$

Even though (69) and (70) generically are different, the corresponding squared momenta $P_{L,R}^2$ in (64) and (67) are the same. Moreover, the corresponding momentum states are the same once we make the appropriate identification

$$n^i = \epsilon^{ij} m_j \ , \tag{71}$$

where ϵ^{ij} is a unit antisymmetric 2×2 matrix (that is, $\epsilon^{12} = -\epsilon^{21} = +1$ or -1). Thus, at the self-dual point (with half-integer B-field) D7-branes satisfying the "no winding flow" condition have the same spectrum as D9-branes satisfying the condition (62), and *not* the usual "no momentum flow" condition (60). At first this might seem a bit strange, as we can ask what is the analogous statement for D9-branes that satisfy the usual "no momentum flow condition" (60). The answer to this question, actually, is very simple. It is not difficult to show that D9-branes that satisfy the usual "no momentum flow" condition have the same spectrum as D7-branes satisfying the "no winding flow" condition at the self-dual point in the moduli space of Kähler structures corresponding to $\det(G) = 1$ and $B_{12} = 0$.

Now everything falls in place, and we arrive at the following consistent picture. D-branes transverse to a torus are the same whether the B-field on the torus is zero or non-zero. However, D-branes wrapped on a torus are sensitive to whether the B-field is zero or non-zero. In the former case they are T-dual to the corresponding D-branes transverse to the torus even at generic points in the moduli space of Kähler structures, and at the self-dual point without the B-field the two types of D-branes are indistinguishable provided that they satisfy the usual "no momentum flow" and "no winding flow" conditions, respectively. In the case with non-zero B-field there is no T-duality between the two types of D-branes (in the presence of orientifold planes) at generic points of the Kähler structure moduli space as the T-duality S-transformation would map a torus with half-integer B-field into

a torus with non-zero B-field which does not take half-integer values. At the self-dual point with half-integer B-field the S-transformation is consistent with the orientifold action, but precisely at this point D-branes transverse to the torus and D-branes wrapped on the torus are indistinguishable, except that the former still satisfy the usual "no winding flow" condition, while the latter satisfy the modified constraint (62). Here it is crucial that the latter type of D-branes *cannot* be consistently coupled to the orientifold planes of the same spatial dimensionality (but, at the self-dual point, couple consistently to the corresponding O-planes transverse to the torus as they are indistinguishable from the corresponding D-branes transverse to the torus). Note that above we arrived at these conclusions in the case of T^2, where we have two inequivalent self-dual points in the Kähler structure moduli space. The generalization to higher tori should be evident once we observe that even though the number of inequivalent self-dual points grows, the latter always have *half*-integer (or zero) B-field, so that different self-dual points correspond to different half-integer B-field configurations. The above discussion should make it evident that the geometric picture of orientifolds with B-flux we described in this section is completely consistent with T-duality considerations. In particular, the fact that D-branes transverse to a torus with B-flux behave quite differently from D-branes wrapped on such a torus is no longer mysterious.

Finally, let us ask to what extent T-duality can be useful in the usual sense. In particular, suppose we have D-branes wrapped on a small volume (in the string units) torus with non-zero B-field, and we would like to perform a T-duality transformation that maps this torus to a large volume torus. Is there such a T-duality transformation? This question has already been answered in [24,34,32], but we would like to reiterate this point here for the sake of completeness. Here we will be a bit more general, and closely follow the discussion in [32]. Thus, consider D9-branes wrapped on T^2 with the metric G_{ij} and B-field $B_{ij} = B_{12}\Sigma$, where $B_{12} = 1/k$, $k \in \mathbf{N} - \{1\}$. Now consider the T-duality transformation (54) with

$$\alpha = I \ , \quad \beta = 0 \ , \quad \gamma = k\Sigma \ , \quad \delta = I \ . \tag{72}$$

We will denote this T-duality transformation by P. The corresponding matrix $E' = G' + B'$ is given by

$$E' = -BG^{-1}B - B = \frac{B_{12}^2}{\det(G)}\ G - B \ . \tag{73}$$

Thus, the T-duality transformation P amounts to

$$G \to G' = \frac{B_{12}^2}{\det(G)}\ G \ , \quad B \to B' = -B \ . \tag{74}$$

In particular, a small volume torus with B-flux is mapped to a large volume torus with opposite B-flux. (Note that $\det(G') = B_{12}^4/\det(G)$.)

Next, let us see what happens to D9-branes under the T-duality transformation P. It is not difficult to see that the transformation P can be written as

$$P = ST^kS \ . \tag{75}$$

Under the first S-transformation D9-branes are mapped to D7-branes, the subsequent T-transformations do not affect the dimensionality of the branes, and the last S-transformation

maps D7-branes back to D9-branes. Thus, the T-duality transformation P maps Dp-branes wrapped on a small volume T^2 with B-flux ($B_{12} = 1/k$) to Dp-branes wrapped on a large volume T^2 with opposite B-flux[19]. (Generalizations to higher tori should be clear.)

III. K3 ORIENTIFOLDS WITH B-FLUX

Having understood toroidal orientifolds with B-flux, we would like to discuss K3 orientifolds with B-flux next. The setup of this section is mostly[20] going to be the Ω orientifold[21] of Type IIB on $\mathbf{R}^{1,5} \otimes \mathrm{K3}$, where $\mathrm{K3} = T^4/\mathbf{Z}_M$. Here the orbifold group generator g acts as $gz_1 = \omega z_1$, $gz_2 = \omega^{-1} z_2$, where $z_{1,2}$ are the complex coordinates parametrizing T^4, and $\omega \equiv \exp(2\pi i/M)$. Note that in order for the orbifold action to be crystallographic, M must be 2,3,4 or 6. The resulting background has $\mathcal{N} = 1$ supersymmetry in six dimensions, and contains D9-branes for $M = 3$, and both D9- and D5-branes for $M = 2, 4, 6$. K3 orientifolds without B-flux have been studied in detail in [3]. The cases with rank $b = 2, 4$ B-flux in the directions of K3 were discussed in [5]. Here we would like to revisit K3 orientifolds with B-flux using our improved understanding of their underlying geometric structure. In subsections A, B, C, D we will discuss the $\mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_6, \mathbf{Z}_4$ models, respectively. In subsection E we briefly summarize the results of this section.

A. The $\mathbf{Z}_2$ Models

Let us first consider the Ω orientifold of Type IIB on $\mathbf{R}^{1,5} \otimes (T^4/\mathbf{Z}_2)$, where the generator R of $\mathbf{Z}_2$ acts as $Rz_{1,2} = -z_{1,2}$ on the complex coordinates parametrizing T^4. In fact, for our purposes here it will suffice to consider $T^4 = T^2 \otimes T^2$, where the first and the second 2-tori are parametrized by z_1 and z_2, respectively.

The orientifold group is $\mathcal{O} = \{1, R, \Omega, \Omega R\}$. Here and in the following $\Omega = \Omega_-$, that is, Ω induces the SO type of orientifold projection on D9-branes. Note that the presence of Ω among the orientifold group elements implies that we have the O9$^-$-plane. Suppose the B-field in the compact directions is trivial. Then the presence of the ΩR orientifold group element implies that we have 16 O5$^-$-planes as well. This has a non-trivial implication on the action of the $\mathbf{Z}_2$ orbifold group element R on the D9- and D5-brane Chan-Paton charges. Let $\gamma_{R,9}$ and $\gamma_{R,5}$ be the corresponding Chan-Paton matrices. (Here for definiteness we will assume that all D5-branes are placed at the same O5$^-$-plane located at the origin $z_1 = z_2 = 0$.) Then we can show that (up to equivalent representations) [2]

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[19] In fact, this was used in [32] in the discussion of unification via Kaluza-Klein thresholds in gauge theories compactified on non-commutative tori, that is, tori with non-zero B-flux.

[20] In the $\mathbf{Z}_3$ cases we will also consider ΩR orientifolds, where R reverses the sign of all coordinates on K3. In such orientifolds we have D5-branes but no D9-branes.

[21] In the $\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_6$ cases Ω will actually be accompanied by an additional action as in [3] - see subsections B,C,D for details.

$$\gamma_{R,9} = \gamma_{R,5} = i\sigma_3 \otimes I_{16} \ . \tag{76}$$

To show this, let us first note that the presence of the O9$^-$-plane implies that we must introduce 32 D9-branes to cancel the 10-form R-R charge. Similarly, the presence of 16 O5$^-$-planes implies that we must introduce 32 D5-branes to cancel the 6-form R-R charge. Thus, all Chan-Paton matrices will be 32×32 dimensional. Next, the matrix $\gamma_{\Omega,9}$ is symmetric [2]:

$$\gamma^T_{\Omega,9} = +\gamma_{\Omega,9} \ . \tag{77}$$

This follows from the fact that the orientifold projection on D9-branes is of the SO type. (Note that in this case we always include the appropriate minus sign in the definition of the Möbius strip amplitude as in (17).) On the other hand, $\gamma_{\Omega,5}$ is antisymmetric [2]:

$$\gamma^T_{\Omega,5} = -\gamma_{\Omega,5} \ . \tag{78}$$

This follows from the fact that the Ω projection on D5-branes must be of the Sp type [2] (which is consistent with [35]). One way to see this is to consider the action of Ω^2 in the 59 sector, where $\Omega^2 = -1$ (note that in the 99 and 55 sectors $\Omega^2 = +1$) [2]. On the other hand, the orientifold 5-planes are of the O5$^-$ type. This implies that $\gamma_{\Omega R,5}$ must be symmetric:

$$\gamma^T_{\Omega R,5} = +\gamma_{\Omega R,5} \ . \tag{79}$$

From (78) and (79) together with the fact that we must have (the choice of the sign is immaterial)

$$\gamma_{\Omega R,5} = \pm\gamma_{\Omega,5}\gamma_{R,5} \ , \tag{80}$$

we obtain

$$\gamma^T_{R,5} = -\gamma_{\Omega,5}\gamma_{R,5}\gamma_{\Omega,5} \ . \tag{81}$$

Now consider the basis where $\gamma_{R,5}$ is diagonal. Then $\gamma^T_{R,5} = +\gamma_{R,5}$, which together with (81) and the fact that we must have

$$\gamma^2_{\Omega,5} = \gamma^2_{\Omega R,5} = 1 \ , \tag{82}$$

implies that

$$\gamma^2_{R,5} = -1 \ . \tag{83}$$

This implies that the eigenvalues of the matrix $\gamma_{R,5}$ are $\pm i$. For the $\mathbf{Z}_2$ orbifold projection in the 59 open string sector to be consistent (that is, so that we get only $\mathbf{Z}_2$ valued phases from the action of the orbifold group on the Chan-Paton charges in the 59 sector), we must then require that the eigenvalues of the $\gamma_{R,9}$ matrix must also be $\pm i$, so that

$$\gamma^2_{R,9} = -1 \ . \tag{84}$$

Note that this is consistent with the analog of (78) for the D9-branes, namely,

$$\gamma_{\Omega R,9}^T = -\gamma_{\Omega R,9} \ . \tag{85}$$

Finally, the twisted tadpole cancellation condition implies that (in the aforementioned setup) the matrices $\gamma_{R,9}$ and $\gamma_{R,5}$ are traceless [1,2]. This then implies (76).

Note that in the above discussion we have assumed that the O5-planes are of the O5$^-$ type. The physical reason for this is clear - had we assumed that the O5-planes were of the O5$^+$ type, we would not have been able to cancel all tadpoles by introducing D5-branes. Another important point is that the twisted Chan-Paton matrices $\gamma_{R,9}$ and $\gamma_{R,5}$ both must have eigenvalues $\pm i$ or ± 1, which, as we have explained above, follows from the requirement that the $\mathbf{Z}_2$ orbifold projection be consistent in the 59 sector. In the above model (with trivial B-field) we must choose these matrices to have eigenvalues $\pm i$ - had we chosen $\gamma_{R,5}$ with eigenvalues ± 1, we would have found that the O5-planes are of the O5$^+$ type. In fact, the constraint (84) can be alternatively derived by considering the Möbius strip amplitude in this model. Thus, we can organize the latter into four terms according to the Chan-Paton factors that multiply them:

$$\begin{aligned} \mathcal{M} = -\left(\frac{1}{2}\right)^2 \Big[& \mathrm{Tr}(\gamma_{\Omega,9}^{-1}\gamma_{\Omega,9}^T)\mathcal{Z}(\Omega, 9) + \mathrm{Tr}(\gamma_{\Omega R,9}^{-1}\gamma_{\Omega R,9}^T)\mathcal{Z}(\Omega R, 9) + \\ & \mathrm{Tr}(\gamma_{\Omega R,5}^{-1}\gamma_{\Omega R,5}^T)\mathcal{Z}(\Omega R, 5) + \mathrm{Tr}(\gamma_{\Omega,5}^{-1}\gamma_{\Omega,5}^T)\mathcal{Z}(\Omega, 5) \Big] \ , \end{aligned} \tag{86}$$

where the overall factor of $(1/2)^2$ arises due to the orientifold and orbifold projections. Let us discuss each term in the above Möbius strip amplitude. The first term containing $\mathcal{Z}(\Omega, 9)$ when rewritten in the closed string tree-channel corresponds to the closed string exchange between D9-branes and the O9$^-$-plane. In fact, (77) is precisely the statement that the O9-plane is of the O9$^-$ type (note the overall minus sign in the definition of the Möbius strip amplitude in (86)). Similarly, the term containing $\mathcal{Z}(\Omega R, 5)$ when rewritten in the closed string tree-channel corresponds to the closed string exchange between D5-branes and the O5-planes. Moreover, the characters $\mathcal{Z}(\Omega, 9)$ and $\mathcal{Z}(\Omega R, 5)$ are actually identical except for the corresponding momentum respectively winding sums (that is, the string oscillator contributions to these characters are identical). Note that (79) is the statement that the O5-planes are of the O5$^-$ type. Next, consider the term containing $\mathcal{Z}(\Omega R, 9)$. The latter corresponds to the 99 sector Möbius contribution with the $\mathbf{Z}_2$ orbifold projection inserted on the boundary. Similarly, the term containing $\mathcal{Z}(\Omega, 5)$ corresponds to the 55 Möbius contribution with the $\mathbf{Z}_2$ orbifold projection inserted on the boundary. In fact, the characters $\mathcal{Z}(\Omega R, 9)$ and $\mathcal{Z}(\Omega, 5)$ are actually identical[22]. Putting

[22]These characters actually vanish. More precisely, all the characters vanish by supersymmetry. However, the bosonic and fermionic pieces in the aforementioned characters vanish separately. Thus, in the open string loop-channel the NS and R contributions to these characters vanish separately. Moreover, in the closed string tree-channel the NS-NS and R-R contributions to these characters also vanish separately. This, in particular, implies that the corresponding terms in the Möbius strip amplitude do not contribute to massless tadpoles. In fact, these terms are often dropped when discussing this model (as in, *e.g.*, [2]). However, these terms are important to keep when discussing the spectrum and vertex operators in this model. In particular, one has to make

all of the above together, we can now derive a non-trivial constraint on $\gamma_{R,9}$. Thus, note that the traces $\mathrm{Tr}(\gamma^{-1}_{\Omega,9}\gamma^T_{\Omega,9}) = \mathrm{Tr}(\gamma_I) = 32$ and $\mathrm{Tr}(\gamma^{-1}_{\Omega R,5}\gamma^T_{\Omega R,5}) = \mathrm{Tr}(\gamma_I) = 32$ in front of the characters $\mathcal{Z}(\Omega, 9)$ respectively $\mathcal{Z}(\Omega R, 5)$ are identical. This implies that the traces $\mathrm{Tr}(\gamma^{-1}_{\Omega R,9}\gamma^T_{\Omega R,9}) = \mathrm{Tr}(\gamma^2_{R,9})$ and $\mathrm{Tr}(\gamma^{-1}_{\Omega,5}\gamma^T_{\Omega,5}) = -\mathrm{Tr}(\gamma_I) = -32$ in front of the characters $\mathcal{Z}(\Omega R, 9)$ respectively $\mathcal{Z}(\Omega, 5)$ must also be identical for the $\mathbf{Z}_2$ orbifold projection in the 99 sector to be consistent with that in the 55 sector. This then implies (84). As we will see in the following, requiring consistency of the orbifold projection in the Möbius strip amplitude will result in additional non-trivial constraints in the models with non-zero B-flux.

Before we consider the cases with non-zero B-flux, for later convenience let us review the $\mathbf{Z}_2$ model of [1,2] without the B-field. The closed string sector contains the six dimensional $\mathcal{N} = 1$ supergravity multiplet, one untwisted (self-dual) tensor supermultiplet, 4 untwisted hypermultiplets, and 16 twisted hypermultiplets. Note that the 16 fixed points of the $\mathbf{Z}_2$ orbifold give rise to hypermultiplets but no (anti-self-dual) tensor multiplets as all 16 fixed points are even under the orientifold action. This follows from the fact that all 16 orientifold 5-planes located at the fixed points are of the $\mathrm{O5}^-$ type. Next, let us discuss the open string spectrum. The gauge group is $U(16)_{99} \otimes U(16)_{55}$, and the massless matter consists of the following hypermultiplets:

$$2 \times (\mathbf{120}; \mathbf{1})_{99} \ , \tag{87}$$

$$2 \times (\mathbf{1}; \mathbf{120})_{55} \ , \tag{88}$$

$$(\mathbf{16}; \mathbf{16})_{95} \ . \tag{89}$$

Here semi-colon separates the 99 and 55 gauge quantum numbers. Note that $U(1)$'s are actually anomalous, and are broken via the generalized Green-Schwarz mechanism [36,37], so that the gauge group is actually $SU(16)_{99} \otimes SU(16)_{55}$.

Here we would like to stress one important point. In particular, the Ω projection on D5-branes is of the Sp type, while the O5-planes located at the 16 $\mathbf{Z}_2$ fixed points are of the $\mathrm{O5}^-$ type, that is, they induce the SO type of orientifold projection on the Chan-Paton charges of D5-branes. The above discussion relates this to the fact that the latter projection is determined by $\gamma_{\Omega R,5}$, and *not* by $\gamma_{\Omega,5}$. Nonetheless, at first it might appear a bit strange that the projection on D5-branes is of the SO type - from the arguments of [35] we would expect (subgroups of) symplectic gauge groups coming form D5-branes. In particular, appearance of antisymmetric representations (namely, $\mathbf{120}$ of $SU(16)$) in the 55 sector might seem puzzling in the context of [35]. However, there is no puzzle here as there is a geometric explanation of this point. To arrive at this explanation, however, we will first need to review the difference between the $\mathrm{O5}^-$- and $\mathrm{O5}^+$-planes. Here we will be closely following the discussion in [38].

Thus, consider Type IIB on $\mathbf{R}^{1,9}$ in the presence of an O5-plane. The O5-plane is located at the fixed point at the origin of $\mathbf{R}^4/\mathbf{Z}_2$, where the $\mathbf{Z}_2$ action simultaneously reflects all four coordinates of $\mathbf{R}^4$ transverse to the O5-plane. The orientifold replaces a 3-sphere $\mathbf{S}^3$ around the origin of $\mathbf{R}^4$ by $\mathbf{RP}^3 = \mathbf{S}^3/\mathbf{Z}_2$. (Recall that the real projective n-space $\mathbf{RP}^n$

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sure that the $\mathbf{Z}_2$ orbifold projection is consistent, which is precisely the constraint we are going to discuss in a moment.

is defined as the quotient $\mathbf{S}^n/\mathbf{Z}_2$ of the n-sphere $\mathbf{S}^n$ defined via $\sum_{i=1}^{n+1} x_i^2 = \rho^2$, where the action of $\mathbf{Z}_2$ on the coordinates x_i, $i = 1, \dots, n+1$, is given by $x_i \to -x_i$, and ρ is the radius of the n-sphere $\mathbf{S}^n$.) Now consider unorientable closed world-sheets $\Sigma = \mathbf{RP}^2$. Such a world-sheet is embeddable in $\mathbf{RP}^3$. Thus, we can define a $\mathbf{Z}_2$ charge for the O5-plane as follows. Consider a constant NS-NS B-flux. Then the world-sheets $\Sigma = \mathbf{RP}^2$ contribute to the path integral with an extra phase

$$\exp\left(i\int_\Sigma B\right) , \tag{90}$$

which is $+1$ for the trivial B-flux, and -1 for the half-integer B-flux (recall that the B-flux is quantized in the presence of an orientifold plane). The aforementioned $\mathbf{Z}_2$ charge assignments are then as follows. In the former case we assign charge 0, while in the latter case we assign charge 1, and the $\mathbf{Z}_2$ charge is defined modulo 2. One can then show that the O5-plane with the $\mathbf{Z}_2$ charge 0 is of the O5$^-$ type, while the one with the charge 1 is of the O5$^+$ type [38]. This, for instance, can be seen by considering the following BPS configuration with eight supercharges. Let the O5-plane fill the coordinates $x_0, x_1, x_2, x_3, x_4, x_5$ of $\mathbf{R}^{1,9}$, and a $\frac{1}{2}$NS5-brane[23] fill the coordinates $x_0, x_1, x_2, x_3, x_4, x_6$. That is, the O5-plane and the $\frac{1}{2}$NS5-plane intersect at 90 degrees in the 56-plane. Now consider the origin of the 789 space transverse to both the O5-plane and the $\frac{1}{2}$NS5-brane. The orientifold replaces a 2-sphere around the origin of this space by $\mathbf{RP}^2$. The NS-NS B-flux couples magnetically to NS5-branes. Thus, with the appropriate normalization $\int_{\mathbf{RP}^2} B$ counts the number of $\frac{1}{2}$NS5-branes modulo 2. This implies that the aforementioned $\mathbf{Z}_2$ charge of the O5-plane, which with this normalization can be identified with $\int_{\mathbf{RP}^2} B$, changes by 1 every time the O5-plane crosses the $\frac{1}{2}$NS5-brane[24]. To see that the O5$^-$-plane has the $\mathbf{Z}_2$ charge 0, consider the T-dual version of the above setup, where we have an O3-plane. Then from Montonen-Olive self-duality of the $SO(2k)$ gauge theories we conclude that the O3$^-$-plane must have zero $\mathbf{Z}_2$ charge or else it would not be invariant under the $SL(2,\mathbf{Z})$ symmetry of Type IIB.

Now we can explain why the O5-planes are of the O5$^-$ type in the aforementioned model of [1,2] without the B-field. Thus, naively we expect that the O5-planes are of the O5$^+$ type as the orientifold projection is of the SO type on D9-branes implying that the orientifold projection is of the Sp type on the D5-branes. This would require that the corresponding $\mathbf{Z}_2$ charge related to the B-field for the orientifold planes is 1, that is, we have an odd-half-integer B-flux in $\mathbf{RP}^2$. However, the O5-planes are located at the $\mathbf{Z}_2$ orbifold fixed points,

[23] In our conventions a $\frac{1}{2}$NS5-brane is the S-dual of a D5-brane, whose R-R charge is $+1$, while O5$^\pm$-planes have the R-R charges ± 2, respectively. Note that an NS5-brane is S-dual of a pair of D5-branes, which combine into a dynamical 5-brane - in the presence of an O5-plane D5-branes always move in pairs.

[24] Here we note that for an Op-plane with $p \leq 5$ one can define another $\mathbf{Z}_2$ charge (in the appropriate normalization) via $\int_{\mathbf{RP}^{5-p}} C^{(5-p)}$ [38], where $C^{(5-p)}$ is a Ramond-Ramond form. The relevant brane configuration here is that of an Op-plane intersecting with a D$(p+2)$-brane such that we have 8 unbroken supercharges. The aforementioned $\mathbf{Z}_2$ charge then changes by 1 every time the Op-plane crosses the D$(p+2)$-brane. This additional $\mathbf{Z}_2$ charge, however, will not be important in the subsequent discussions.

and each $\mathbf{Z}_2$ orbifold fixed point corresponds to a collapsed 2-sphere $\mathbf{P}^1$. As was pointed out in [39], in the *conformal field theory* $\mathbf{Z}_2$ orbifold there is an odd-half-integer *twisted* B-field[25] stuck inside of each collapsed $\mathbf{P}^1$. The orientifold replaces these $\mathbf{P}^1$'s by $\mathbf{RP}^2$'s, so that we have an odd-half-integer twisted B-flux stuck inside of each collapsed $\mathbf{RP}^2$. This additional twisted B-flux converts the would-be O5$^+$-planes into O5$^-$-planes, which is the result we wished to explain.

Next, let us consider the Ω orientifold of Type IIB on $T^4/\mathbf{Z}_2$ in the presence of the B-field. Here we will mainly focus on the case of $T^4 = T^2 \otimes T^2$ with B-field of rank $b = 2$ turned on in the directions of the first T^2 (while the B-field in the directions of the second T^2 is trivial). Generalizations to generic T^4's with B-field of rank $b = 2$ as well as $b = 4$ should be evident.

In the case of B-field of rank b we have $n_{f-} = 8(1 + 1/2^{b/2})$ O5$^-$-planes and $n_{f+} = 8(1 - 1/2^{b/2})$ O5$^+$-planes. In the closed string $\mathbf{Z}_2$ twisted sector the orientifold projection at the $\mathbf{Z}_2$ orbifold fixed points where the O5$^-$-planes are located gives rise to hypermultiplets, while at the fixed points where the O5$^+$-planes are located it gives rise to (anti-self-dual) tensor multiplets. On the other hand, the untwisted closed string sectors of the models with $b = 2, 4$ are the same as that of the $b = 0$ model. Thus, the closed string sector of the model with B-field of rank b contains the six dimensional $\mathcal{N} = 1$ supergravity multiplet, one untwisted (self-dual) tensor supermultiplet, 4 untwisted hypermultiplets, n_{f-} twisted hypermultiplets and n_{f+} twisted (anti-self-dual) tensor multiplets.

Next, let us discuss the open string sector. As we have already mentioned, we will specialize to the case of $b = 2$. First, let us understand the geometric structure of the orientifold planes. Let the vielbeins on the first T^2 (where we have non-zero B-flux) be e_i, $i = 1, 2$, while the vielbeins on the second T^2 (where the B-flux is trivial) be d_i, $i = 1, 2$. Let us define $e_3 \equiv -e_1 - e_2$, and $d_3 \equiv -d_1 - d_2$. The 16 $\mathbf{Z}_2$ fixed points are located at $(0, 0), (e_a/2, 0), (0, d_a/2), (e_a/2, d_b/2)$, $a, b = 1, 2, 3$. Without loss of generality we can choose the following distribution for the O5-planes. At $(0, 0), (0, d_a/2), (e_i/2, 0), (e_i/2, d_a/2)$ we have 12 O5$^-$-planes, while at $(e_3/2, 0), (e_3/2, d_a/2)$ we have 4 O5$^+$-planes.

At first it might seem that the above setup is completely consistent, in particular, that there is no difficulty with having O5$^-$- and O5$^+$-planes at the same time. Thus, this is certainly consistent in the toroidal case, so it might seem that in the $T^4/\mathbf{Z}_2$ orbifold case this should also be consistent. However, there is a subtlety here. In particular, note that the orientifold projection is of the SO type on D9-branes, and, therefore, it is of the Sp type on D5-branes. The odd-half-integer twisted B-flux is present at all 16 fixed points of the $T^4/\mathbf{Z}_2$ orbifold. Thus, repeating the above argument we would conclude that all 16 O5-planes must be of the O5$^-$ type. So there seems to be a puzzle here. Let us, therefore, try to understand this point better.

To begin with, let us note that there are two separate issues here. First, by studying the orientifold action in the closed string sector we unambiguously arrive at the conclusion that we have 12 O5$^-$- and 4 O5$^+$-planes as in the toroidal case. However, this does not guarantee that, once we introduce *both* 32 D9-branes and 16 D5-branes, the orientifold action in the

[25] The twisted B-field plays an important role in the context of orientifolds in a number of setups - see, *e.g.*, [40] and [12].

open string sector is indeed consistent. This statement might seem surprising at first as the orientifold action in the 99 and 55 sectors certainly seems to be consistent. This follows from our analyses of the corresponding toroidal orientifolds with B-flux (and we do not expect any obstruction in consistently orbifolding the 99 and 55 sectors). However, the subtlety here can (and, as we will point out in a moment, does) arise in the 59 sector. In particular, it is not always sufficient to consider, say, the Klein bottle amplitude (and the corresponding tadpoles) to conclude that we have some numbers of Op^-- and Op^+-planes. Rather, we must also make sure that in any given setup these objects indeed induce the SO and Sp type of orientifold projections on Dp-branes (that is, we must make sure that the couplings between the Op-planes and Dp-branes are consistent).

Since the issue we are discussing here appears to be subtle, we would like to proceed step-by-step. Thus, first let us make sure that the orientifold projection in the closed string sector is indeed such that we have n_{f-} twisted hypermultiplets and n_{f+} twisted tensor multiplets. Thus, we must show that the orientifold action on n_{f+} fixed points has an extra minus sign compared with that on the other n_{f-} fixed points. One way to see this explicitly was already discussed in [5]. Thus, let us consider a special point in the moduli space of T^4's corresponding to the $SO(8)$ symmetry. At this point the B-field has rank $b = 2$ [5], and in the conformal field theory we have the $[SO(8)]_L \otimes [SO(8)]_R$ current algebra at level 1 on the closed string world-sheet. The $\mathbf{Z}_2$ orbifold action reduces this current algebra to $[SU(2)^4]_L \otimes [SU(2)^4]_R$ at level 1. The vertex operators for the 16 fixed points are especially simple at this point. In particular, they carry the following quantum numbers under $[SU(2)^4]_L \otimes [SU(2)^4]_R$: $(\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}||\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1})$, plus states obtained by simultaneous permutations of the $SU(2)$ factors for both left- and right-movers. The orientifold action reduces the $[SU(2)^4]_L \otimes [SU(2)^4]_R$ current algebra to $[SU(2)^4]_{\rm diag}$, and the fixed points now carry the following quantum numbers: $(\mathbf{3}_s \oplus \mathbf{1}_a, \mathbf{1}, \mathbf{1}, \mathbf{1})$, plus states obtained by permutations of the $SU(2)$ factors. Note that the states transforming in $\mathbf{3}_s$'s are symmetric under Ω, while the states transforming in $\mathbf{1}_a$'s are antisymmetric. Thus, we indeed have 12 twisted hypermultiplets and 4 twisted tensor multiplets in this case[26]. In particular, this is consistent with the fact that we have n_{f-} O5$^-$-planes located at the fixed points giving rise to twisted hypermultiplets, and n_{f+} O5$^+$-planes located at the fixed points giving rise to twisted tensor multiplets (as can be explicitly seen by considering the Klein bottle amplitude in this model).

Next, let us discuss the open string sector. In the 99 sector we have Chan-Paton matrices $\gamma_{S_a,9}$ corresponding to non-commuting Wilson lines[27], and also the twisted Chan-Paton matrix $\gamma_{R,9}$. We can write the corresponding Chan-Paton matrices as follows:

$$\gamma_{S_a,9} = \gamma_a \otimes I_2 \otimes I_8 \ , \tag{91}$$

[26]It is not difficult to see that in the $b = 4$ case we then have 10 twisted hypermultiplets and 6 twisted tensor multiplets [5].

[27]As in the previous section, here we are going to view half-integer B-flux on the first T^2 with vielbeins e_i in terms of the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold (with discrete torsion) of the torus $\widetilde{T}^2$ with vielbeins $E_i = 2e_i$ and zero B-flux. The non-commuting Wilson lines then correspond to half-lattice shifts $S_i X_i = X_i + E_i/2$ on $\widetilde{T}^2$ acting on the Chan-Paton degrees of freedom.

$$\gamma_{R,9} = I_2 \otimes \nu\sigma_3 \otimes I_8 \ , \tag{92}$$

where γ_a are the 2×2 matrices corresponding to the D_4 or D_4' type of Wilson lines given in (28) respectively (29). On the other hand, for the twisted Chan-Paton matrix $\gamma_{R,9}$ *a priori* we have two inequivalent choices: $\nu^2 = +1$ and $\nu^2 = -1$. However, as we will show in a moment, these choices are correlated with the choices of the non-commuting Wilson lines as follows:

$$D_4: \quad \nu^2 = +1 \ , \tag{93}$$

$$D_4': \quad \nu^2 = -1 \ . \tag{94}$$

To show this, let us consider the Möbius strip amplitude:

$$\mathcal{M} = -\left(\frac{1}{2}\right)^2 \Big[\mathcal{Y}(\Omega, 9) + \mathcal{Y}(\Omega R, 9) + \mathcal{Y}(\Omega R, 5) + \mathcal{Y}(\Omega, 5)\Big] \ . \tag{95}$$

Here the characters $\mathcal{Y}$, which contain the corresponding Chan-Paton factors, are defined as follows. The character $\mathcal{Y}(\Omega, 9)$ corresponds to the 99 sector contribution with the identity element of the $\mathbf{Z}_2$ orbifold group inserted on the boundary of the Möbius strip; the character $\mathcal{Y}(\Omega R, 9)$ corresponds to the 99 sector contribution with element R of the $\mathbf{Z}_2$ orbifold group inserted on the boundary. Similarly, the character $\mathcal{Y}(\Omega R, 5)$ corresponds to the 55 sector contribution with the identity element inserted on the boundary; the character $\mathcal{Y}(\Omega, 5)$ corresponds to the 55 sector contribution with the element R inserted on the boundary. Starting from (17), it is not difficult to see that the character $\mathcal{Y}(\Omega R, 9)$ is given by:

$$\begin{aligned}\mathcal{Y}(\Omega R, 9) &= \frac{1}{4}\mathcal{Z}(\Omega R, 9)\Big[\mathrm{Tr}(\gamma^{-1}_{\Omega R,9}\gamma^T_{\Omega R,9}) + \sum_{a=1}^{3} \mathrm{Tr}(\gamma^{-1}_{\Omega R S_a,9}\gamma^T_{\Omega R S_a,9})\Big] \\ &= \frac{1}{4}\mathcal{Z}(\Omega R, 9)\mathrm{Tr}(\gamma^2_{R,9})\Big[1 + \sum_{a=1}^{3} \eta_{aa}\Big] \ ,\end{aligned} \tag{96}$$

where the character $\mathcal{Z}(\Omega R, 9)$ here is the same as in (86). Next, let us consider the character $\mathcal{Y}(\Omega, 5)$. Since there are no Wilson lines in the 55 sector, we have

$$\mathcal{Y}(\Omega, 5) = \mathrm{Tr}(\gamma^{-1}_{\Omega,5}\gamma^T_{\Omega,5})\mathcal{Z}(\Omega, 5) \ , \tag{97}$$

where the character $\mathcal{Z}(\Omega, 5)$ is the same as in (86). Now repeating the argument after (86) for the above model, we conclude that in the Möbius strip amplitude for the $\mathbf{Z}_2$ orbifold projection in the 99 sector to be consistent with that in the 55 sector, the following constraint must be satisfied:

$$\frac{1}{4}\mathrm{Tr}(\gamma^2_{R,9})\Big[1 + \sum_{a=1}^{3} \eta_{aa}\Big] = \mathrm{Tr}(\gamma^{-1}_{\Omega,5}\gamma^T_{\Omega,5}) \ . \tag{98}$$

We can simplify this constraint as follows. First, note that $\mathrm{Tr}(\gamma^{-1}_{\Omega,5}\gamma^T_{\Omega,5}) = -16$. Also, $\mathrm{Tr}(\gamma^2_{R,9}) = 32\nu^2$. Finally, $\sum_{a=1}^3 \eta_{aa} = +1$ for the D_4' type of Wilson lines (29), and $\sum_{a=1}^3 \eta_{aa} = -3$ for the D_4 type of Wilson lines (28). Putting all of this together, we conclude that for the D_4 type of Wilson lines we must have $\nu^2 = +1$, whereas for the D_4' type of Wilson lines we must have $\nu^2 = -1$.

The constraint we have just derived is the analogue of the corresponding constraint in the case without B-flux. In the latter case, as we have already discussed above, relaxing this constraint would result in a model where tadpoles/anomalies cannot be canceled completely. In the case with non-zero B-field relaxing the corresponding constraint would also result in inconsistent models. Thus, for instance, consider taking the D_4 type of Wilson lines with $\nu^2 = -1$. The resulting model has the following massless spectrum. We have already discussed the closed string sector (which is independent of the choice of the Chan-Paton matrices) - it contains 16 hypermultiplets and 5 tensor multiplets (for $b = 2$). In the open string sector the gauge group is $[U(8) \otimes \mathbf{Z}_2]_{99} \otimes U(8)_{55}$, and the massless matter is given by the following hypermultiplets:

$$2 \times (\mathbf{36}; \mathbf{1})_{99} \ , \tag{99}$$

$$2 \times (\mathbf{1}; \mathbf{36})_{55} \ , \tag{100}$$

$$(\mathbf{8}_+; \mathbf{8})_{95} \ , \tag{101}$$

$$(\mathbf{8}_-; \mathbf{8})_{59} \ , \tag{102}$$

where the subscript $\pm$ in the 95 sector refers to the 99 $\mathbf{Z}_2$ charge. Note that this spectrum is anomalous. In particular, the irreducible R^4 gravitational anomaly does not cancel. A similar conclusion applies to the case where the Wilson lines are of the D_4' type, while the $\mathbf{Z}_2$ twisted Chan-Paton matrix is chosen such that $\nu^2 = +1$.

Next, we would like to study the models arising for the aforementioned two inequivalent choices satisfying the above consistency condition. To begin with, let us consider the following choice:

$$\gamma_{S_a,9} = \gamma_a \otimes I_2 \otimes I_8 \ , \tag{103}$$

$$\gamma_{R,9} = I_2 \otimes i\sigma_3 \otimes I_8 \ , \tag{104}$$

where the γ_a matrices correspond to the D_4' type of Wilson lines given in (29). In the 55 sector we must also specify the choice of $\gamma_{R,5}$, which is a 16×16 matrix (and not a 32×32 matrix as we have only 16 D5-branes). The choice of this matrix must be consistent with the orientifold action. In particular, let us consider an O5$^-$-plane, say, that located at the fixed point at the origin $(0,0)$. Then, if we place 16 D5-branes on top of this orientifold plane, according to the above arguments we must choose $\gamma_{R,5}$ such that its eigenvalues are $\pm i$. On the other hand, suppose we consider an O5$^+$-plane, say, that located at the fixed point $(e_3/2, 0)$. Then, if we placed D5-branes on top of this O5$^+$-plane, we would encounter an inconsistency. To see this let us repeat the argument given in the beginning of this subsection for the case where the O5-plane is of the O5$^+$ type. Thus, in this case we have

$$\gamma^T_{\Omega R,5} = -\gamma_{\Omega R,5} \ . \tag{105}$$

Together with (78) and (80) this implies that

$$\gamma^T_{R,5} = +\gamma_{\Omega,5}\gamma_{R,5}\gamma_{\Omega,5} \ . \tag{106}$$

Now consider the basis where $\gamma_{R,5}$ is diagonal. Then $\gamma^T_{R,5} = +\gamma_{R,5}$, which together with (106) and (82) implies that

$$\gamma_{R,5}^2 = +1 \ . \tag{107}$$

This implies that the eigenvalues of the matrix $\gamma_{R,5}$ are ± 1. Note, however, that the eigenvalues of the matrix $\gamma_{R,9}$ are $\pm i$. That is, in the 59 sector the $\mathbf{Z}_2$ orbifold action is inconsistent as the 59 states would have phases $\pm i$ (instead of ± 1). At first it might seem that this difficulty can be simply avoided by concluding that in this background it is inconsistent to place D5-branes at the O5$^+$-planes, albeit it is consistent to place them at the O5$^-$-planes. However, this naive resolution fails due to the following simple argument[28] - there is no obstruction to moving D5-branes from an O5$^-$-plane to an O5$^+$-plane in this model.

Thus, let us consider the $T^4/\mathbf{Z}_2$ model with B-field of rank $b = 2$. Let us place all 16 D5-branes at the O5$^-$-plane located at the origin $(0,0)$. Then the consistent choice of $\gamma_{R,5}$ is (up to equivalent representations) given by

$$\gamma_{R,5} = i\sigma_3 \otimes I_8 \ . \tag{108}$$

The gauge group of this model is $[U(8) \otimes \mathbf{Z}_2]_{99} \otimes U(8)_{55}$. The open string massless matter consists of the following hypermultiplets

$$2 \times (\mathbf{28}; \mathbf{1})_{99} \ , \tag{109}$$
$$2 \times (\mathbf{1}; \mathbf{28})_{55} \ , \tag{110}$$
$$(\mathbf{8}_+; \mathbf{8})_{95} \ , \tag{111}$$
$$(\mathbf{8}_-; \mathbf{8})_{95} \ . \tag{112}$$

Here the subscript $\pm$ in the 95 sector refers to the 99 $\mathbf{Z}_2$ discrete gauge charge of the corresponding state. Note that $U(1)$'s are actually anomalous, and are broken by the generalized Green-Schwarz mechanism [41]. In particular, the states that participate in the generalized Green-Schwarz mechanism are (certain linear combinations of) the R-R scalars in the twisted hypermultiplets. Thus, the gauge group is actually $[SU(8) \otimes \mathbf{Z}_2]_{99} \otimes SU(8)_{55}$. Moreover, the corresponding *two* twisted hypermultiplets are eaten in the process of Higgsing $U(1)$'s. Note that the above spectrum is free of the irreducible R^4 and F^4 anomalies.

Note that we can Higgs the 55 gauge group by giving VEVs to the hypermultiplets $2 \times (\mathbf{1}; \mathbf{28})_{55}$. Thus, for instance, we can break the 55 gauge group $SU(8)_{55}$ down to $Sp(8)_{55}$ by giving appropriate VEVs to the aforementioned hypermultiplets - under the breaking $SU(8) \rightarrow Sp(8)$, the antisymmetric representation $\mathbf{28}$ of $SU(8)$ decomposes as $\mathbf{28} = \mathbf{1} + \mathbf{27}$ in terms of the $Sp(8)$ representations. Actually, we can start from the $U(8)_{55}$ gauge group with the anomalous $U(1)_{55}$ factor, and break $U(8)_{55}$ down to $Sp(8)_{55}$. In this process 28 out of the 56 hypermultiplets $2 \times (\mathbf{1}; \mathbf{28})_{55}$ are eaten in the Higgs mechanism[29]. The leftover 28 hypermultiplets transform in $\mathbf{1} \oplus \mathbf{27}$ of $Sp(8)_{55}$. Note that this Higgsing corresponds to nothing but moving together all 16 D5-branes off the O5$^-$-plane into the bulk. The VEV of

[28] This effective field theory argument was pointed out to me by Alex Buchel.

[29] Note that a singlet of $Sp(8)_{55}$ is eaten in Higgsing the $U(1)_{55}$ factor. This implies that only one (instead of two) of the closed twisted hypermultiplets is eaten, namely, in the process of Higgsing the anomalous $U(1)_{99}$ factor.

the leftover singlet hypermultiplet then corresponds to the location of D5-branes in the bulk (note that there are 4 real scalars in a hypermultiplet, and those in the singlet hypermultiplet parametrize the location of D5-branes in four real dimensions of K3). On the other hand, we can further break the $Sp(8)$ gauge group by giving a VEV to the leftover hypermultiplet which is in **27** of $Sp(8)$. This Higgsing corresponds to pulling D5-branes apart from each other in groups of 4 (or multiples thereof) - each dynamical 5-brane, which consists of 4 D5-branes[30], separately gives rise to an $Sp(2)$ gauge group. On the other hand, k coincident dynamical 5-branes (corresponding to $4k$ coincident D5-branes) give rise to $Sp(2k)$ gauge group, with all four dynamical 5-branes coincident giving rise to the $Sp(8)$ gauge symmetry.

Even though the general case is straightforward to discuss, from now on it will suffice for our purposes here to consider moving together all 16 D5-branes off the O5$^-$-plane, so that in the bulk they give rise to the $Sp(8)$ gauge symmetry. In this subspace of the moduli space the gauge group is $[SU(8)\otimes \mathbf{Z}_2]_{99}\otimes Sp(8)_{55}$, and the open string massless matter consists of the following hypermultiplets:

$$2\times(\mathbf{28};\mathbf{1})_{99}\ , \tag{113}$$
$$(\mathbf{1};\mathbf{1}\oplus\mathbf{27})_{55}\ , \tag{114}$$
$$(\mathbf{8}_+;\mathbf{8})_{95}\ , \tag{115}$$
$$(\mathbf{8}_-;\mathbf{8})_{95}\ . \tag{116}$$

In fact, this spectrum can be derived directly in the orientifold language if we consider placing 16 D5-branes in the bulk (that is, away from the $\mathbf{Z}_2$ orbifold fixed points). In particular, let us place 8 D5-branes at a generic point (z_1,z_2) in the bulk. Then we must place the other 8 D5-branes at the point $(-z_1,-z_2)$, so that the entire background is invariant under the action of the ΩR element of the orientifold group. If we did not have to perform the further $\mathbf{Z}_2$ orbifold projection with respect to the element R, the 55 gauge group at such a generic point would be $U(8)_{55}$. However, the $\mathbf{Z}_2$ orbifold projection reduces $U(8)_{55}$ to its subgroup $Sp(8)_{55}$. Note that the reason why the rank of the unbroken gauge group $Sp(8)_{55}$ is halved compared with that of $U(8)_{55}$ is that $\gamma_{\Omega R,5}$ and $\gamma_{R,5}$ do not commute, in fact, they anticommute as can be seen from the relation $\gamma_{R,5}=-\gamma_{\Omega,5}\gamma_{R,5}\gamma_{\Omega,5}$, which holds in the basis where $\gamma_{R,5}$ is diagonal. Furthermore, the 55 gauge bosons are in the symmetric representation (so that the gauge group is symplectic), while the 55 hypermultiplets are in the antisymmetric representation (which is reducible for symplectic gauge groups). The latter fact is due to the extra minus sign that the $\mathbf{Z}_2$ twist R has when acting on the hypermultiplets compared with when it acts on the gauge bosons.

Next, we would like to ask what would happen if we bring D5-branes to one of the O5$^+$-planes. Here we expect that the 55 gauge symmetry should be enhanced, and, moreover, the 55 gauge group must be a subgroup of $Sp(16)$. The latter is the gauge group we would obtain in the toroidal case, and in the orbifold case the 55 gauge group would have to be determined by the choice of $\gamma_{R,5}$. Thus, if we choose $\gamma_{R,5}$ to be

$$\gamma_{R,5}=\sigma_3\otimes I_8\ , \tag{117}$$

[30]Here two pairings take place - one due to the orientifold projection, and the other one due to the orbifold projection.

then the gauge group coming from 16 D5-branes on top of an O5$^+$-plane is $Sp(8) \otimes Sp(8)$. Thus, the gauge group of the model at this point in the moduli space is $[SU(8) \otimes \mathbf{Z}_2]_{99} \otimes [Sp(8) \otimes Sp(8)]_{55}$, and the open string massless matter is given by:

$$2 \times (\mathbf{28}; \mathbf{1}, \mathbf{1})_{99} \ , \tag{118}$$

$$(\mathbf{1}; \mathbf{8}, \mathbf{8})_{55} \ , \tag{119}$$

$$\frac{1}{2}(\mathbf{8}_+; \mathbf{8}, \mathbf{1})_{95} \ , \tag{120}$$

$$\frac{1}{2}(\mathbf{8}_+; \mathbf{1}, \mathbf{8})_{95} \ , \tag{121}$$

$$\frac{1}{2}(\mathbf{8}_-; \mathbf{8}, \mathbf{1})_{95} \ , \tag{122}$$

$$\frac{1}{2}(\mathbf{8}_-; \mathbf{1}, \mathbf{8})_{95} \ . \tag{123}$$

Note that the 99 spectrum is the same as before, and the 55 spectrum can be deduced by keeping the states in the toroidal compactification invariant under the $\mathbf{Z}_2$ orbifold action. The states in the 95 sector, however, cannot be deduced in this way as the $\mathbf{Z}_2$ action in this sector is inconsistent - recall that the eigenvalues of $\gamma_{R,9}$ are $\pm i$, while the eigenvalues of $\gamma_{R,5}$ are ± 1. The above 95 spectrum has been written down as follows. First, the number of 95 hypermultiplets must be the same whether D5-branes are on top of the O5$^+$-plane or in the bulk. Second, the two $Sp(8)$ subgroups in the 55 sector are on the equal footing, so that the spectrum should possess a symmetry under the permutation of these two subgroups. This then fixes the spectrum as above. This spectrum, however, is *anomalous*. Indeed, it contains half-hypermultiplets in complex representations. Note that had we not distinguished the 99 $\mathbf{Z}_2$ discrete gauge quantum numbers, naively we might have thought that we have one hypermultiplet in $(\mathbf{8}; \mathbf{8}, \mathbf{1})$, and one hypermultiplet in $(\mathbf{8}; \mathbf{1}, \mathbf{8})$ of $SU(8)_{99} \otimes [Sp(8) \otimes Sp(8)]_{55}$. This is, however, not the case, and, as we see, the 99 $\mathbf{Z}_2$ discrete gauge symmetry does indeed play an important role.

Thus, we have arrived at an inconsistency - we started from a seemingly consistent setup where we had all 16 D5-branes on top of an O5$^-$-plane, and then moved them to one of the O5$^+$-planes. Even though in the bulk the 55 gauge theory appears to be consistent, at the O5$^+$-plane it is anomalous. We have seen this in the effective field theory language, and in the orientifold language this inconsistency is translated to that in the $\mathbf{Z}_2$ orbifold action in the 59 sector due to the fact that $\gamma_{R,5}$ is given by (117), while $\gamma_{R,9}$ is given by (104). Here we can ask if we could possibly have chosen $\gamma_{R,5}$ as in (108) even if D5-branes are on top of an O5$^+$-plane. It is, however, not difficult to see that this choice is inconsistent. Thus, we know from our discussion which led to (107) that the choice consistent with having D5-branes on top of an O5$^+$-plane is that given in (117), and not in (108). However, we can readily see what goes wrong if we make this inconsistent choice in the language of the effective field theory as well. Thus, let us assume for a moment that $\gamma_{R,5}$ is given by (108) even though D5-branes are placed on top of an O5$^+$-plane. Then the gauge group of the model would be $[U(8) \otimes \mathbf{Z}_2]_{99} \otimes U(8)_{55}$, and the massless open string spectrum would read (here we have kept anomalous $U(1)$'s for the convenience reasons that will become clear in a moment):

$$2 \times (\mathbf{28}; \mathbf{1})_{99} \ , \tag{124}$$

$$2 \times (\mathbf{1}; \mathbf{36})_{55} \ , \tag{125}$$

$$(\mathbf{8}_+; \mathbf{8})_{95} \ , \tag{126}$$

$$(\mathbf{8}_-; \mathbf{8})_{95} \ . \tag{127}$$

Note that the 95 part of this spectrum now looks consistent. However, this spectrum is actually anomalous once we take into account the massless content of the closed string sector, which consists of the six dimensional $\mathcal{N}=1$ supergravity multiplet, 5 tensor supermultiplets, and 16 hypermultiplets. Thus, for instance, the R^4 gravitational anomaly does not cancel in this model. This is due to the fact that we have 55 hypermultiplets in **36** (symmetric) representation of $SU(8)$ instead of **28** (antisymmetric), the reason being that D5-branes now are at an $\mathrm{O5}^+$-plane instead of an $\mathrm{O5}^-$-plane. Let us mention that we have kept the anomalous $U(1)$ factors merely for the counting convenience - eventually they are broken, so we can drop them, but then instead of 16 closed string hypermultiplets we would have only 14 as two twisted hypermultiplets (or, more precisely, certain linear combinations thereof) are eaten in the corresponding Higgs mechanism.

Thus, the assumption that we can have both $\mathrm{O5}^-$- and $\mathrm{O5}^+$-planes at various fixed points of the conformal field theory orbifold $T^4/\mathbf{Z}_2$ does not seem to be self-consistent. This in accord with our discussion of the role of the twisted B-flux in the collapsed $\mathbf{P}^1$'s at the orbifold fixed points. In particular, according to this discussion in the case of the conformal field theory orbifold $T^4/\mathbf{Z}_2$ we expect that only $\mathrm{O5}^-$-planes can be placed at the orbifold fixed points. The obstruction for placing $\mathrm{O5}^+$-planes at the fixed points comes precisely from the fact that we have twisted B-flux. This, in turn, gives us a hint of how to possibly remedy the situation in the case of the orientifold of Type IIB on $T^2/\mathbf{Z}_2$ with B-flux. More precisely, we can attempt to guess what the correct background for such an orientifold might be (the orientifold of the conformal field theory orbifold $T^4/\mathbf{Z}_2$ does not appear to be the correct background).

The key observation here is that to have an $\mathrm{O5}^+$-plane at the orbifold fixed point we must have *trivial* twisted B-flux in the corresponding collapsed $\mathbf{P}^1$. However, if we simply turn off the twisted B-flux, the corresponding background could no longer be described within the world-sheet (that is, the conformal field theory) approach. Indeed, in this case we have a true geometric $\mathbf{A}_1$ singularity at each of the fixed points with the twisted B-field turned off. To avoid this, we would have to *blow up* the orbifold singularity by giving a VEV to the corresponding twisted hypermultiplet. Note, however, that if we first consider the $T^4/\mathbf{Z}_2$ orbifold with collapsed $\mathbf{P}^1$'s and then orientifold, the fixed points where we have $\mathrm{O5}^+$-planes would give rise to twisted tensor multiplets only, so such a blow-up would not be possible. However, we can proceed as follows. Consider Type IIB on K3, where K3 is defined as follows. Consider the $T^4/\mathbf{Z}_2$ orbifold with B-flux of rank $b=2$. At the 12 fixed points where we expect $\mathrm{O5}^-$-planes we have collapsed $\mathbf{P}^1$'s with half-integer twisted B-flux, and these points locally can be described in the conformal field theory. At the other 4 fixed points where we expect $\mathrm{O5}^+$-planes we have $\mathbf{P}^1$'s of non-zero size but with trivial twisted B-flux. Moreover, let us assume that the size of K3 is large compared with the size of these blow-ups. Then we can indeed locally describe the other 12 fixed points in the exactly solvable conformal field theory language of the $\mathbf{C}^2/\mathbf{Z}_2$ orbifold. On the other hand, the 4 blown-up fixed points without the twisted B-flux can no longer be described in terms of an exactly solvable orbifold conformal field theory (albeit there should exist some conformal field

theory description of such a K3 which is not exactly solvable but corresponds to some non-trivial sigma-model). Next, start from Type IIB on the K3 surface we have just described, and consider its Ω orientifold. At the 12 fixed points with the twisted B-flux we have O5$^-$-planes, and these fixed points give rise to twisted hypermultiplets, while at the 4 fixed points without the twisted B-flux we have O5$^+$-planes, and these fixed points give rise to twisted tensor multiplets. Since there are no twisted hypermultiplets at these 4 fixed points, we cannot blow down the corresponding $\mathbf{P}^1$'s. In this sense, the orientifolding procedure does not commute with the blowing-up procedure. This, in turn, might signal that there could be a caveat in the above discussion. In particular, it is not completely evident that the boundary states at the 4 blown-up fixed points, if such are at all present, carry the correct R-R charges to be interpreted as O5$^+$-planes. In fact, technical issues in conformal field theories with boundaries might, at least partially, be responsible for difficulties in making this point quantitatively more precise. Nonetheless, we can attempt to proceed further in understanding the underlying qualitative picture assuming that the boundary states at the blown-up fixed points indeed correspond to the O5$^+$-planes.

What can we say about this orientifold? With the aforementioned assumption, it is reasonable to assume that if we place 16 D5-branes at one of the O5$^-$-planes, then we get the spectrum described above, which (modulo the missing 99 $\mathbf{Z}_2$ discrete symmetry) is the same as that given in [5]. This spectrum is consistent, and with the aforementioned interpretation of the K3 background might adequately describe physics at the corresponding point in the moduli space. Once we move D5-branes off the O5$^-$-plane, the 55 gauge symmetry in the bulk is $Sp(8)$ (or an appropriate subgroup thereof). The key question, however, is what happens when we approach an O5$^+$-plane - after all this was where we have encountered trouble in the above discussion in the context of the conformal field theory orbifold. Note, however, that in the case of the K3 surface under consideration the moduli space corresponding to the motion of D5-branes is no longer flat as it was in the case of the conformal field theory orbifold (more precisely, in the latter case it is flat everywhere except for the fixed points). Because of the non-zero size of the corresponding $\mathbf{P}^1$'s, D5-branes actually might not be able to come on top of the O5$^+$-planes (whose locations in K3 are given by points inside of the blown-up $\mathbf{P}^1$'s). If so, this would avoid the contradiction we have encountered in the conformal field theory orbifold case. That is, the 55 gauge symmetry would never be enhanced to $Sp(8)\otimes Sp(8)$ - in the bulk it is at most $Sp(8)$, at an O5$^-$-plane it is $SU(8)$, while the points corresponding to D5-branes being on top of O5$^+$-planes might possibly be thought of as being at infinite distance in the moduli space. If so, the aforementioned K3 surface might indeed be the correct consistent background for the corresponding orientifold with B-flux. Once again, however, it is not completely clear how to make the above discussion quantitatively more precise due to the fact that the conformal field theory corresponding to such a K3 would not be exactly solvable[31].

Before we summarize the findings of this subsection, we would like to give two further pieces of evidence that in considering the Ω orientifold of Type IIB on the conformal field

[31]In the next section we will discuss four dimensional orientifolds with $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetry where the aforementioned difficulties are avoided within exactly solvable conformal field theory compactifications.

theory $T^4/\mathbf{Z}_2$ orbifold with B-flux one indeed runs into various subtle inconsistencies. The first piece of such evidence comes from studying the 99 sector moduli space in the above model. The second piece of evidence comes from considering the second seemingly consistent setup, namely, that with the D_4 type of Wilson lines accompanied with the twisted Chan-Paton matrix $\gamma_{R,9}$ with $\nu^2 = +1$.

Thus, so far we have focused on the 55 sector moduli space. Similar considerations apply to the 99 sector. Thus, suppose we start from the point in the moduli space where all 16 D5-branes are placed at an O5$^-$-plane. At this point we have $[SU(8) \otimes \mathbf{Z}_2]_{99} \otimes SU(8)_{55}$ gauge symmetry. We can Higgs the $SU(8)_{99}$ gauge group down to $Sp(8)_{99}$ or its appropriate subgroups, and this Higgsing now corresponds to turning on Wilson lines. We can describe these Wilson lines as in (30), except that now we also have the twisted Chan-Paton matrix $\gamma_{R,9}$. Thus, consider the following choice of $\gamma_{S_a,9}$ and $\gamma_{R,9}$:

$$\gamma_{S_1,9} = \sigma_3 \otimes \rho(a) \otimes I_8 \ , \quad (128)$$

$$\gamma_{S_2,9} = \sigma_1 \otimes \rho(a^{-1}) \otimes I_8 \ , \quad (129)$$

$$\gamma_{S_3,9} = i\sigma_2 \otimes I_2 \otimes I_8 \ , \quad (130)$$

$$\gamma_{R,9} = I_2 \otimes i\sigma_1 \otimes I_8 \ . \quad (131)$$

Here $\rho(a) \equiv \mathrm{diag}(a, a^{-1})$, where a is a complex phase. For definiteness we have chosen $\gamma_{S_3,9} = +\gamma_{S_1,9}\gamma_{S_2,9}$. Note that $\gamma_{R,9}$ commutes with $\gamma_{S_3,9}$. However, for $\gamma_{S_i,9}$, $i = 1, 2$, we have:

$$\gamma_{R,9}\gamma_{S_i,9} = \gamma^{-1}_{S_i,9}\gamma_{R,9} \ . \quad (132)$$

Note that for $a^2 = 1$ $\gamma_{R,9}$ actually commutes with $\gamma_{S_i,9}$.

Before we proceed further, the following remarks are in order. First, for $a^2 = 1$ (as well as $a^2 = -1$) the aforementioned Wilson lines can be thought of in terms of the freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold (with discrete torsion). However, at generic points they correspond to more general Wilson lines that do not possess a simple (freely acting) orbifold description. However, at rational values of the phase, say, $a = \exp(2\pi i/k)$, $k \in \mathbf{Z}$, such an orbifold description does exist. Thus, for instance, let $k \in 2\mathbf{N}+1$. Then $\gamma^2_{S_1,9}$ generates a $\mathbf{Z}_k$ discrete group. In the language of a freely acting orbifold this can be understood as having $\mathbf{Z}_k$ valued shifts of the torus lattice. Note that such a shift is mapped to its inverse by the reflection of the torus coordinates described by the twist R. This is precisely the reason why we have chosen the Chan-Paton matrices $\gamma_{S_a,9}$ and $\gamma_{R,9}$ so that they satisfy the conjugation relation (132). In fact, this conjugation relation should also hold for generic (that is, irrational) phases as well. To see this, consider a non-trivial Wilson line (which we schematically write as a diagonal $N \times N$ matrix)

$$W = \exp\left(i \int_C A\right) \ , \quad (133)$$

where $A = \mathrm{diag}(\theta_1, \ldots, \theta_N)$ is a constant gauge field, and C is a non-trivial 1-cycle. Now consider a $\mathbf{Z}_2$ orbifold action such that it reverses the coordinate parametrizing C. Then the action of the $\mathbf{Z}_2$ orbifold generator R on the Wilson line is given by (note that $R^2 = 1$)

$$RWR = W^{-1} \ , \quad (134)$$

which follows from the fact that A reverses its sign under the action of R. This then implies that the conjugation relation (132) also holds for generic values of the phase a.

Next, note that at, say, $a = 1$ the above choice of $\gamma_{S_a,9}$ and $\gamma_{R,9}$ is equivalent to that in (103) and (104). In particular, the 99 gauge group is $[SU(8) \otimes \mathbf{Z}_2]_{99}$, while the 55 gauge group is $SU(8)_{55}$, and all the D5-branes are placed at an O5$^-$-plane. Now let us continuously deform the above Wilson lines from the point $a = 1$ to, say, the point $a = i$. At this point the Wilson lines $\gamma_{S_a,9}$ are of the D_4 type (while the Wilson lines are of the D'_4 type at $a = 1$) - the eigenvalues of all three matrices $\gamma_{S_a,9}$ are $\pm i$. So at this point the 99 gauge group would be $Sp(16) \otimes \mathbf{Z}_2$ if we did not have a further $\mathbf{Z}_2$ orbifold action. Let us see what the gauge group is once we perform the $\mathbf{Z}_2$ orbifold projection with respect to $\gamma_{R,9}$. Actually, $\gamma_{R,9}$ does not commute with $\gamma_{S_i,9}$, $i = 1, 2$, at $a = i$, in fact, they anticommute. Thus, we should find a basis where the twisted Chan-Paton matrix commutes with the Wilson lines. Such a basis is given by $\gamma_{S_a,9}$ together with $\gamma_{\widetilde{R},9}$, where $\widetilde{R} \equiv RS_3$, and we can choose

$$\gamma_{\widetilde{R},9} = \gamma_{R,9}\gamma_{S_3,9} = i\sigma_2 \otimes i\sigma_1 \otimes I_8 \ . \tag{135}$$

Note that the eigenvalues of $\gamma_{\widetilde{R},9}$ are no longer $\pm i$ but ± 1, which is consistent with the fact that for the D_4 type of Wilson lines we must have $\nu^2 = +1$. This then implies that the 99 gauge group is actually $[Sp(8) \otimes Sp(8) \otimes \mathbf{Z}_2]_{99}$. On the other hand, the 55 gauge group is unchanged - it is still $SU(8)_{55}$. It is then not difficult to see that here we are running into a problem similar to that in the case where we attempted to place D5-branes on top of an O5$^+$-plane - the resulting spectrum in the 59 sector is anomalous. The reason for this is that the $\mathbf{Z}_2$ projection in the 99 sector corresponds to the twisted Chan-Paton matrix $\gamma_{\widetilde{R},9}$ with eigenvalues ± 1, while that in the 55 sector corresponds to $\gamma_{R,5}$, whose eigenvalues are $\pm i$. That is, here we have the same type of inconsistency in the 59 sector as that encountered in the case of D5-branes sitting on top of an O5$^+$-plane. Note that if we attempt to interpret $\gamma_{R,9}$ (instead of $\gamma_{\widetilde{R},9}$) as the twisted Chan-Paton matrix with respect to which we must perform the $\mathbf{Z}_2$ orbifold projection, then the latter might naively appear to be consistent in the 59 sector. However, it is not difficult to see that having performed the $\mathbf{Z}_2$ orbifold projection in the 59 sector with respect to $\gamma_{R,9}$ (together with $\gamma_{R,5}$), the resulting 59 sector states would *not* transform in representations of the $[Sp(8) \otimes Sp(8) \otimes \mathbf{Z}_2]_{55}$ gauge group, which is due to the fact that $\gamma_{R,9}$ and $\gamma_{S_i,9}$ do not commute at this point in the moduli space[32].

Note that the aforementioned inconsistency arises if we choose the background to be the Ω orientifold of Type IIB on the conformal field theory orbifold $T^4/\mathbf{Z}_2$ as in this case there is no obstruction to continuously deforming the above Wilson lines from the point $a = 1$ to the point $a = i$. However, if we consider the partially blown-up K3 surface described above as the consistent orientifold background, such a continuous deformation of Wilson lines might no longer be possible as the corresponding moduli space is no longer flat. As in the discussion of the 55 moduli space, however, it is not clear how to make this point quantitatively precise. At any rate, if the aforementioned K3 surface is indeed a consistent background for the

[32]We will encounter similar situations in other models in the following subsections, where we will discuss this point in more detail.

above orientifold, there would have to exist an obstruction to continuously deforming the Wilson lines from the points with $a^2=+1$ to those with $a^2=-1$, as well as an obstruction to placing D5-branes on top of an O5$^+$-plane. If this is indeed the case, then the above $b=2$ model (where the Wilson lines are of the D_4' type, $\nu^2=-1$, and all 16 D5-branes are placed at an O5$^-$-plane) would be consistent. In fact, then we could also consider its $b=4$ counterpart as follows. Consider the Ω orientifold of Type IIB on K3, where K3 is a $T^4/\mathbf{Z}_2$ orbifold with $b=4$ B-flux, and 10 of the $\mathbf{Z}_2$ orbifold fixed points locally can be described in the language of the conformal field theory orbifold $\mathbf{C}^2/\mathbf{Z}_2$, while the other 6 fixed points are blown-up, and the corresponding twisted B-flux is trivial. The former 10 fixed points are those at which we have O5$^-$-planes, while the latter 6 fixed points are those at which we have O5$^+$-planes. The $b=4$ B-flux can be described as follows. For the sake of simplicity consider $T^4=T^2\otimes T^2$. On the first T^2 we have non-commuting Wilson lines $\gamma_{S_i,9}$, while on the second T^2 we have non-commuting Wilson lines $\gamma_{T_i,9}$ (these two sets of Wilson lines, however, commute). Now consider the case where both sets of Wilson lines are of the D_4' or D_4 type. Then the consistent choice for the twisted Chan-Paton matrix $\gamma_{R,9}$ is that with $\nu^2=-1$. The twisted Chan-Paton matrix $\gamma_{R,5}$ (which is an 8×8 matrix as we have 8 D5-branes in this case) is then also fixed. If we place all D5-branes at an O5$^-$-plane, then the gauge group is $[U(4)\otimes\mathbf{Z}_2\otimes\mathbf{Z}_2]_{99}\otimes U(4)_{55}$, and the open string sector massless hypermultiplets are given by:

$$2\times(\mathbf{6};\mathbf{1})_{99}\ , \tag{136}$$
$$2\times(\mathbf{1};\mathbf{6})_{55}\ , \tag{137}$$
$$(\mathbf{4}_{++};\mathbf{4})_{95}\ , \tag{138}$$
$$(\mathbf{4}_{+-};\mathbf{4})_{95}\ , \tag{139}$$
$$(\mathbf{4}_{-+};\mathbf{4})_{95}\ , \tag{140}$$
$$(\mathbf{4}_{--};\mathbf{4})_{95}\ , \tag{141}$$

where the subscript $\pm\pm$ in the 95 sector refers to the 99 $\mathbf{Z}_2\otimes\mathbf{Z}_2$ discrete gauge charges. This spectrum (together with that from the closed string sector) is free of the irreducible R^4 and F^4 anomalies. The $U(1)$ factors are anomalous as usual, and are broken via the generalized Green-Schwarz mechanism. Note that (modulo the missing 99 $\mathbf{Z}_2\otimes\mathbf{Z}_2$ discrete gauge symmetry) this is the spectrum of the $b=4$ $\mathbf{Z}_2$ model in [5].

Next, in the $b=2$ case, we would like to discuss the second *a priori* consistent choice for the Wilson lines and the twisted Chan-Paton matrix, namely, where the Wilson lines are of the D_4 type, and the twisted Chan-Paton matrix is chosen such that $\nu^2=+1$. Note that in this case we must have $\gamma^2_{R,5}=+1$, from which it follows that D5-branes must be placed at an O5$^+$-plane. Note that just as in the previous case the conformal field theory $T^4/\mathbf{Z}_2$ orbifold does not appear to be the correct background for such an orientifold. In particular, in the conformal field theory $T^4/\mathbf{Z}_2$ orbifold case we would have all the troubles we have encountered for the previous choice of the Wilson lines and the twisted Chan-Paton matrix. Thus, for instance, there would be no obstruction to moving D5-branes from an O5$^+$-plane to an O5$^-$-plane, and at the latter point in the moduli space the spectrum of the model would be anomalous. However, in the present case (that is, where the Wilson lines are of the D_4 type, and the twisted Chan-Paton matrix is chosen such that $\nu^2=+1$) there is an additional puzzling issue. In particular, from our discussion of the role of the twisted

B-flux it actually follows that we could not have an O5$^+$-plane at a fixed point (locally) corresponding to the conformal field theory orbifold $\mathbf{C}^2/\mathbf{Z}_2$. In fact, if we can at all have an O5$^+$-plane at a $\mathbf{Z}_2$ fixed point, we expect that the latter would have to be blown-up, and the twisted B-field would have to be trivial. Moreover, in this case it would be impossible to place D5-branes on top of such an O5$^+$-plane.

In the light of the above discussion, we would like to see whether we can find any inconsistency in the model where the Wilson lines are of the D_4 type, the twisted Chan-Paton matrix is chosen such that $\nu^2 = +1$, and all D5-branes are placed at an O5$^+$-plane. It is not difficult to see that the gauge group of this model is $[Sp(8)\otimes Sp(8)\otimes \mathbf{Z}_2]_{99}\otimes[Sp(8)\otimes Sp(8)]_{55}$, and the massless open string hypermultiplets are given by:

$$(\mathbf{8},\mathbf{8};\mathbf{1},\mathbf{1})_{99}\ , \tag{142}$$
$$(\mathbf{1},\mathbf{1};\mathbf{8},\mathbf{8})_{55}\ , \tag{143}$$
$$\frac{1}{2}(\mathbf{8}_+,\mathbf{1};\mathbf{8},\mathbf{1})_{95}\ , \tag{144}$$
$$\frac{1}{2}(\mathbf{8}_-,\mathbf{1};\mathbf{8},\mathbf{1})_{95}\ , \tag{145}$$
$$\frac{1}{2}(\mathbf{1},\mathbf{8}_+;\mathbf{1},\mathbf{8})_{95}\ , \tag{146}$$
$$\frac{1}{2}(\mathbf{1},\mathbf{8}_-;\mathbf{1},\mathbf{8})_{95}\ , \tag{147}$$

where the subscript $\pm$ in the 95 sector refers to the 99 $\mathbf{Z}_2$ discrete gauge charge of the corresponding state. Note that the R^4 gravitational anomaly cancels in this model. However, the above spectrum is not completely anomaly free - it contains half-hypermultiplets in real representations[33]. Note that had we not distinguished the 99 $\mathbf{Z}_2$ discrete gauge quantum numbers, naively we might have thought that in the 95 sector we have one hypermultiplet in $(\mathbf{8},\mathbf{1};\mathbf{8},\mathbf{1})$, and one hypermultiplet in $(\mathbf{1},\mathbf{8};\mathbf{1},\mathbf{8})$, which would give a consistent spectrum. This is, however, not the case here, and, once again, the 99 $\mathbf{Z}_2$ discrete gauge symmetry indeed plays an important role. Thus, the above spectrum with the $[Sp(8)\otimes Sp(8)\otimes \mathbf{Z}_2]_{99}\otimes [Sp(8)\otimes Sp(8)]_{55}$ gauge symmetry does not appear to be completely consistent[34], and, as we have discussed above, this gives additional evidence for the conclusion that we cannot have

[33]Note that the fundamental representation of a symplectic gauge group is pseudoreal. However, here we have bifundamental representations of a product symplectic group, and the former are real.

[34]The $Sp(8)^4$ model with the above spectrum but with missing 99 $\mathbf{Z}_2$ discrete gauge symmetry was originally constructed in [1] via the "rational construction" equivalent to the conformal field theory orbifold $T^4/\mathbf{Z}_2$ at the special point in the moduli space of T^4's corresponding to the $SO(8)$ symmetry. This model was discussed in the context of general $T^4/\mathbf{Z}_2$ compactifications with $b=2$ B-flux in the second reference in [19], and more recently in [27]. However, all of these references missed the importance of the 99 $\mathbf{Z}_2$ discrete gauge symmetry, whose presence, as we have just pointed out, leads to a subtle inconsistency in the model.

an O5$^+$-plane at the conformal field theory $\mathbf{C}^2/\mathbf{Z}_2$ orbifold fixed point[35].

Let us summarize the results of this subsection. We have considered the Ω orientifold of Type IIB on $T^4/\mathbf{Z}_2$ with non-zero B-flux. The latter requires that n_{f+} of the 16 O5-planes be of the O5$^+$ type. However, the latter cannot be placed at conformal field theory orbifold fixed points due to the non-zero twisted B-flux inside of the corresponding collapsed $\mathbf{P}^1$'s (while it is precisely an O5$^-$-plane that can be consistently placed at such a fixed point). A possible way around this would be to blow up the orbifold fixed point and turn off the twisted B-flux. The price one would have to pay for this, however, is that the corresponding K3 no longer corresponds to an exactly solvable conformal field theory. Because of this, it is not entirely clear whether the corresponding models are completely consistent, albeit their massless spectra appear to be.

We have presented various pieces of evidence that attempts to interpret the aforementioned orientifolds in the context of the conformal field theory $T^4/\mathbf{Z}_2$ orbifold with B-flux run into various inconsistencies visible already at the massless level, in particular, in the 59 sector. In fact, here we would like to suggest a simple geometric interpretation of these inconsistencies. Note that a peculiar feature of the aforementioned orientifolds is that we have D9-branes wrapping a torus (or, more precisely, an orbifold thereof) with B-flux together with D5-branes transverse to this torus. On the one hand, the gauge bundles of branes wrapped on such tori lack vector structure as the corresponding Stieffel-Whitney class is non-vanishing. On the other hand, the presence of D9- and D5-branes implies that we have the 59 sector, where the states transform in the bifundamental representations of the gauge group. The latter require non-trivial vector structure which is in conflict with the lack of vector structure for the 99 gauge bundles. However, in the next section we will be able to avoid this difficulty in non-trivial *four* dimensional $\mathcal{N}=2$ and $\mathcal{N}=1$ supersymmetric orientifolds with B-flux.

B. The $\mathbf{Z}_3$ Models

In this subsection we will consider orientifolds of Type IIB on $\mathbf{R}^{1,5}\otimes(T^4/\mathbf{Z}_3)$ with non-zero B-flux. We will denote the generator of the $\mathbf{Z}_3$ orbifold group via θ, whose action on the complex coordinates z_1, z_2 parametrizing T^4 is given by $\theta z_1=\omega z_1$, $\theta z_2=\omega^{-1}z_2$, where $\omega\equiv\exp(2\pi i/3)$. In fact, for our purposes here it will suffice to consider $T^4=T^2\otimes T^2$, where the first and the second 2-tori are parametrized by z_1 and z_2, respectively.

The orientifolds we will discuss here are defined as follows. The orientifold action is given

[35]Note that the above conclusions also apply to the analogous $b=4$ model, which is constructed as follows. Consider the $(T^2\otimes T^2)/\mathbf{Z}_2$ orbifold. The $b=4$ B-field can be described in terms of the two sets of Wilson lines γ_{S_i} and γ_{T_i}. Choose one of these sets to be of the D_4 type, while the other set to be of the D_4' type. Then it is not difficult to show that the consistent choice for the twisted Chan-Paton matrix is such that $\nu^2=+1$. Consequently, we must place D5-branes at an O5$^+$-plane in this model. The gauge group of this model is $[Sp(4)\otimes Sp(4)\otimes\mathbf{Z}_2\otimes\mathbf{Z}_2]_{99}\otimes[Sp(4)\otimes Sp(4)]_{55}$. In the 95 sector we again have half-hypermultiplets in real representations, which are charged non-trivially under the 99 $\mathbf{Z}_2\otimes\mathbf{Z}_2$ discrete gauge symmetry.

by ΩJ, where Ω interchanges the left- and right-movers (and its action is the same as in the smooth K3 case), while $J = J'$, or $J = RJ'$. Here J' acts as follows. Its action on the untwisted sector fields is trivial, however, it interchanges the θ twisted sector with its inverse θ^{-1} twisted sector. Thus, as explained in [26], the $\Omega J'$ orientifold is precisely that discussed in [3]. The geometric meaning of the J' action was discussed in detail in [12]. Next, R is the simultaneous reflection of the coordinates on T^4: $Rz_{1,2} = -z_{1,2}$. Note that in the $\Omega J'$ orientifold we expect 32 D9-branes but no D5-branes, while in the $\Omega RJ'$ orientifold we expect $32/2^{b/2}$ D5-branes but no D9-branes, where, as before, b is the rank of the untwisted NS-NS B-field.

First, let us discuss the $\Omega J'$ orientifold with $b = 2$ (for definiteness we will assume that the B-flux is turned on in the direction of the first T^2). In this case we have 32 D9-branes but no D5-branes. The B-flux can be described in terms of the freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold with discrete torsion acting on the first T^2, whose action on the D9-brane Chan-Paton charges is given by the matrices γ_{S_a}. In fact, we are now going to show that these matrices must be of the D_4 type, that is, it would be inconsistent to choose them of the D'_4 type. This can be seen as follows. First, the first T^2 (as well as the second T^2) must have $\mathbf{Z}_3$ symmetry. This implies that the vielbeins e_i of the first T^2 are rotated by the action of θ as follows:

$$\theta e_1 = e_2 \ , \quad \theta e_2 = e_3 \ , \quad \theta e_3 = e_1 \ , \tag{148}$$

where $e_3 \equiv -e_1 - e_2$. Next, note that S_a are half-lattice shifts in the directions of e_a. This implies that we must have the following relations between the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold groups elements S_a and the $\mathbf{Z}_3$ orbifold group element θ:

$$\theta S_1 \theta^{-1} = S_2 \ , \quad \theta S_2 \theta^{-1} = S_3 \ , \quad \theta S_3 \theta^{-1} = S_1 \ . \tag{149}$$

That is, the orbifold group elements θ and S_a do *not* commute, and, in fact, they generate a non-Abelian *tetrahedral* subgroup of $SU(2)$ (or a double cover thereof)[36]. This then implies that the corresponding Chan-Paton matrices must satisfy the following relations:

$$\gamma_\theta \gamma_{S_1} \gamma_\theta^{-1} = \gamma_{S_2} \ , \quad \gamma_\theta \gamma_{S_2} \gamma_\theta^{-1} = \gamma_{S_3} \ , \quad \gamma_\theta \gamma_{S_3} \gamma_\theta^{-1} = \gamma_{S_1} \ . \tag{150}$$

This, in particular, implies that

$$\gamma^2_{S_1} = \gamma^2_{S_2} = \gamma^2_{S_3} \ . \tag{151}$$

It then follows that the Wilson lines must be of the D_4 (and *not* D'_4) type[37].

Next, let us discuss solutions to the above conditions. Up to equivalent representations we can write them as follows (here we are using the fact that $\gamma_{S_3} = \eta_{12} \gamma_{S_1} \gamma_{S_2}$):

[36]Note that the discrete torsion between the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ elements is compatible with the $\mathbf{Z}_3$ orbifold action.

[37]Recall that $\gamma^2_{S_a} = \eta_{aa} I_{32}$, where all three η_{aa} equal -1 for the D_4 type of Wilson lines, while two of them equal $+1$ and the third one equals -1 for the D'_4 type of Wilson lines.

$$\gamma_{S_1} = i\sigma_3 \otimes I_{16} \ , \tag{152}$$
$$\gamma_{S_2} = i\sigma_2 \otimes I_{16} \ , \tag{153}$$
$$\gamma_{S_3} = i\eta_{12}\sigma_1 \otimes I_{16} \ , \tag{154}$$
$$\gamma_\theta = \xi_\theta \otimes \Gamma_\theta \ , \tag{155}$$

where

$$\xi_\theta \equiv \left(-\frac{1}{2}\right)[I_2 + i\sigma_1 + i\eta_{12}\sigma_2 + i\eta_{12}\sigma_3] \ , \tag{156}$$

and the 16×16 matrix Γ_θ is a diagonal matrix with non-zero entries taking values $1, \omega, \omega^{-1}$. Note that the 2×2 matrix ξ_θ has eigenvalues ω and ω^{-1}, so that the matrix γ_θ has eigenvalues taking values $1, \omega, \omega^{-1}$.

Note that in the above solution we still have to fix the form of the matrix Γ_θ. It is uniquely determined (up to equivalent representations) once we impose twisted tadpole cancellation conditions. The latter can be deduced as follows. Note that the orientifold with $b = 2$ B-flux is the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold of the orientifold without B-flux. The trace of the twisted Chan-Paton matrix γ_θ is then fixed as in the $\mathbf{Z}_3$ model of [3]:

$$\mathrm{Tr}(\gamma_\theta) = 8 \ . \tag{157}$$

Traces of all the other twisted matrices such as $\gamma_{\theta S_a}$ are then fixed unambiguously by the above relations. In particular:

$$\mathrm{Tr}(\gamma_{\theta S_a}) = \mathrm{Tr}(\gamma_\theta \gamma_{S_a}) = -\eta_{12}\mathrm{Tr}(\gamma_\theta) = \eta_{12}\mathrm{Tr}(\Gamma_\theta) \ . \tag{158}$$

Note that $\mathrm{Tr}(\Gamma_\theta) = -\mathrm{Tr}(\gamma_\theta) = -8$. This implies that up to equivalent representations we have

$$\Gamma_\theta = \mathrm{diag}(\omega, \omega^{-1}) \otimes I_8 \ . \tag{159}$$

Note that in the diagonal basis γ_θ can be written as $\gamma_\theta = \mathrm{diag}(1, 1, \omega, \omega^{-1}) \otimes I_8$.

Next, let us determine the massless spectrum of this model. First, let us discuss the closed string spectrum. It contains the six dimensional $\mathcal{N} = 1$ supergravity multiplet, one untwisted tensor multiplet, 2 untwisted hypermultiplets, 9 twisted tensor multiplets and 9 twisted hypermultiplets. The open string spectrum can be determined as follows. First, note that the 2×2 matrices

$$\gamma_1 \equiv i\sigma_3 \ , \quad \gamma_2 \equiv i\sigma_2 \ , \quad \gamma_3 \equiv i\eta_{12}\sigma_1 \ , \quad \gamma_\theta^{(k)} \equiv \omega^k \xi_\theta \tag{160}$$

define three irreducible two dimensional representations of the tetrahedral subgroup $\mathcal{T}$ of $SU(2)$ labeled by the integer $k = 0, 1, 2$. The aforementioned set of matrices $\gamma_{S_a}, \gamma_\theta$ with Γ_θ given by (159) corresponds to taking 8 copies of the two dimensional representation labeled by $k = 1$ together with 8 copies of the two dimensional representation labeled by $k = 2$. The $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold action (that is, the action of the Chan-Paton matrices γ_{S_a}) breaks the original $SO(32)$ gauge group down to its $Sp(16) \otimes \mathbf{Z}_2$ subgroup. The further $\mathbf{Z}_3$ orbifold action (that is, the action of the twisted Chan-Paton matrix γ_θ) breaks this gauge group

down to its $U(8)$ subgroup[38]. In fact, the open string massless spectrum is given by the $\mathcal{N}=1$ gauge supermultiplet in the adjoint of $U(8)$ plus one hypermultiplet in $\mathbf{36}$ of $U(8)$. (Note that the adjoint of $Sp(16)$ decomposes in terms of the $U(8)$ representations as follows: $\mathbf{136}=\mathbf{64}(0)\oplus\mathbf{36}(+2)\oplus\overline{\mathbf{36}}(-2)$, where we have given the $U(1)$ charges in parentheses, and the latter are normalized so that the fundamental of $SU(8)$ has the $U(1)$ charge $+1$.) One can directly derive[39] this spectrum by keeping the states invariant under the action of the non-Abelian tetrahedral group $\mathcal{T}$. In particular, note that the action of the γ_θ matrix on the $Sp(16)$ part of the gauge group, which is given by the 16×16 matrix Γ_θ, breaks $Sp(16)$ down to $U(8)$ as $\Gamma_\theta=\mathrm{diag}(\omega,\omega^{-1})\otimes I_8$. Alternatively, we can use the following trick. The open string partition function $\mathcal{Z}[\mathcal{T}]$ of the full $\mathcal{T}$ orbifold model can be expressed in terms of the partition functions $\mathcal{Z}[\mathbf{Z}_2\otimes\mathbf{Z}_2]$, $\mathcal{Z}[\mathbf{Z}_3]$ and $\mathcal{Z}[1]$ as follows:

$$\mathcal{Z}[\mathcal{T}]=\mathcal{Z}[\mathbf{Z}_3]+\frac{1}{3}\mathcal{Z}[\mathbf{Z}_2\otimes\mathbf{Z}_2]-\frac{1}{3}\mathcal{Z}[1]\ , \qquad (161)$$

where $\mathcal{Z}[1]$ is the partition function of the model corresponding to the toroidal compactification *without* the B-flux (this model has $\mathcal{N}=2$ supersymmetry and $SO(32)$ gauge group), $\mathcal{Z}[\mathbf{Z}_2\otimes\mathbf{Z}_2]$ is the partition function of the model corresponding to the toroidal compactification with $b=2$ B-flux, that is, the $\mathbf{Z}_2\otimes\mathbf{Z}_2$ freely acting orbifold model (this model has $\mathcal{N}=2$ supersymmetry and $Sp(16)\otimes\mathbf{Z}_2$ gauge group), and $\mathcal{Z}[\mathbf{Z}_3]$ is the partition function of the model corresponding to the $T^4/\mathbf{Z}_3$ compactification *without* the B-flux (this is the $\mathbf{Z}_3$ model of [3] with $\mathcal{N}=1$ supersymmetry, $SO(16)\otimes U(8)$ gauge group and massless hypermultiplets in $(\mathbf{16},\mathbf{8})\oplus(\mathbf{1},\mathbf{28})$ of $SO(16)\otimes U(8)$). If we now write the spectra of the $\mathcal{N}=2$ models in the $\mathcal{N}=1$ language, we can then read off the numbers of gauge bosons and massless hypermultiplets in the full $\mathcal{T}$ orbifold model from $\mathcal{Z}[\mathcal{T}]$ defined as above.

Here we note that the above closed plus open string spectrum, which is the same as that given in [5], is free of the irreducible R^4 and F^4 anomalies. The $U(1)$ factor is anomalous as usual, and is broken via the generalized Green-Schwarz mechanism in a way similar to the $\mathbf{Z}_3$ model of [3] without the B-flux.

Next, let us discuss the $\Omega J'$ orientifold model with $b=4$ B-flux. The latter can be described in terms of two $\mathbf{Z}_2\otimes\mathbf{Z}_2$ freely acting orbifolds acting on the first respectively second T^2. We can choose the corresponding Chan-Paton matrices as follows:

$$\gamma_{S_1}=i\sigma_3\otimes I_2\otimes I_8\ , \qquad (162)$$
$$\gamma_{S_2}=i\sigma_2\otimes I_2\otimes I_8\ , \qquad (163)$$
$$\gamma_{S_3}=i\eta_{12}\sigma_1\otimes I_2\otimes I_8\ , \qquad (164)$$
$$\gamma_{T_1}=I_2\otimes i\sigma_3\otimes I_8\ , \qquad (165)$$
$$\gamma_{T_2}=I_2\otimes i\sigma_2\otimes I_8\ , \qquad (166)$$

[38] Here we note that the $\mathbf{Z}_3$ twist breaks the $\mathbf{Z}_2$ discrete subgroup in the product $Sp(16)\otimes\mathbf{Z}_2$. This point will become important in the next subsection, and we will discuss it there in more detail. However, the fate of the $\mathbf{Z}_2$ discrete gauge symmetry will not be important for our purposes here as no massless states carry non-trivial $\mathbf{Z}_2$ charges.

[39] We will give the details of this derivation in the next subsection.

$$\gamma_{T_3} = I_2 \otimes i\eta'_{12}\sigma_1 \otimes I_8 \ , \tag{167}$$

$$\gamma_\theta = \xi_\theta \otimes \xi'_\theta \otimes \widetilde{\Gamma}_\theta \ , \tag{168}$$

where ξ'_θ is given by the the same expression (156) with η'_{12} instead of η_{12}. Note the 4×4 matrix $\xi_\theta \otimes \xi'_\theta$ has eigenvalues $1, 1, \omega, \omega^{-1}$. This then implies that to satisfy the twisted tadpole cancellation condition $\mathrm{Tr}(\gamma_\theta) = 8$, we must take the 8×8 matrix $\widetilde{\Gamma}_\theta$ to be the identity matrix I_8. It is then not difficult to see that the gauge group of this model is $SO(8)$, and we have no massless hypermultiplets in the open string spectrum[40]. In particular, note that the Wilson lines γ_{S_a} and γ_{T_a} break the original $SO(32)$ gauge group down to $SO(8) \otimes (\mathbf{Z}_2)^2$. The action of the $\mathbf{Z}_3$ orbifold group on the $SO(8)$ part of the gauge group is trivial[41] as it is given by $\widetilde{\Gamma}_\theta = I_8$. The closed string spectrum is the same as in the $b = 2$ model. Note that the spectrum of the $b = 4$ model, which is the same as that given in [5], is free of the irreducible R^4 and F^4 anomalies.

Now we would like to discuss the $\Omega R J'$ orientifolds with B-flux. Let us start with the $b = 2$ case. As before, let us assume that the B-field is turned on in the direction of the first T^2. Let e_i be the vielbeins of the first T^2, and d_i be the vielbeins of the second T^2. As before, we will use the notation $e_3 \equiv -e_1 - e_2$, $d_3 \equiv -d_1 - d_2$. In this model we have 16 O5-planes located at the 16 points fixed under the action of the reflection R. These fixed points are located at $(0, 0), (0, d_a/2), (e_a/2, 0), (e_a/2, d_a/2)$. Note that 12 out of these O5-planes must be of the O5$^-$ type, while 4 must be of the O5$^+$ type. Together with the requirement that the entire background be $\mathbf{Z}_3$ symmetric (so that the further $\mathbf{Z}_3$ orbifold is consistent), this uniquely fixes the allowed distribution of O5-planes. Thus, the O5-planes located at the fixed points $(0, 0), (0, d_a/2)$ must be of the O5$^+$ type, while the rest of the O5-planes are of the O5$^-$ type.

In this orientifold we have 16 D5-branes. The latter must be distributed in a $\mathbf{Z}_3$ symmetric fashion. First, let us consider placing all 16 D5-branes on top of the O5$^+$-plane at the origin $(0, 0)$ of $T^4/\mathbf{Z}_3$. Before the $\mathbf{Z}_3$ orbifold projection the gauge group is $Sp(16)$. The action of the $\mathbf{Z}_3$ orbifold on the Chan-Paton charges is given by the twisted 16×16 Chan-Paton matrix γ_θ. The only constraint on this matrix comes from the twisted tadpole cancellation condition, which in this case reads

$$\mathrm{Tr}(\gamma_\theta) = -8 \ . \tag{169}$$

Note that this is the same twisted tadpole cancellation condition as in the case of D9-branes wrapped on $T^4/\mathbf{Z}_3$ without B-field except for the extra minus sign[42]. This minus sign is, in

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[40]Note that in the diagonal basis the 16×16 matrix $\xi'_\theta \otimes \widetilde{\Gamma}_\theta$ can be written as $\mathrm{diag}(\omega, \omega^{-1}) \otimes I_8$, which is consistent with Γ_θ given in (159).

[41]As in the $b = 2$ case, however, it breaks the $(\mathbf{Z}_2)^2$ discrete gauge symmetry.

[42]Here we note that generally twisted tadpole cancellation conditions for D5-branes transverse to the $\mathbf{C}^2/\mathbf{Z}_M$ orbifold are different from those for D9-branes with the $\mathbf{C}^2/\mathbf{Z}_M$ orbifold in their world-volumes. However, for a particular case of the $\mathbf{Z}_3$ orbifold group they actually coincide with the overall sign depending on the type of the corresponding orientifold plane.

fact, due to the O5-plane here being of the O5$^+$ type (while the twisted tadpole cancellation condition in the case of D9-branes was derived in [3] for the configuration involving the O9$^-$-plane). The above tadpole cancellation condition uniquely fixes the twisted Chan-Paton matrix (up to equivalent representations):

$$\gamma_\theta = \mathrm{diag}(\omega, \omega^{-1}) \otimes I_8 \ . \tag{170}$$

Thus, just as in the case of $b = 2$ $\Omega J'$ orientifold, this model with D5-branes has $U(8)$ gauge group and one massless hypermultiplet in **36** of $U(8)$. (The closed string sector is the same as in the $\Omega J'$ orientifold.) Note that if we attempted to place some D5-branes in the bulk or at other O5-planes, we would not have been able to cancel twisted tadpoles. Indeed, D5-branes away from the origin must be placed in a $\mathbf{Z}_3$ symmetric fashion, that is, the number of D5-branes away from the origin must be a multiple of 3. These branes are then permuted by the action of the $\mathbf{Z}_3$ orbifold, and the corresponding part of the twisted Chan-Paton matrix γ_θ is traceless[43]. The part corresponding to the D5-branes at the origin then cannot have trace equal -8, so that the twisted tadpole cancellation conditions cannot be satisfied. The point we have just discussed can be understood in terms of the effective field theory language as follows. Note that moving D5-branes off the O5$^+$-plane (located at the origin) into the bulk would correspond to giving the appropriate VEV to a massless hypermultiplet. The only massless hypermultiplet in this model (with all D5-branes at the origin) is in **36** of $U(8)$. If we could give a VEV to this hypermultiplet, it would break $U(8)$ down to $SO(8)$. (Note that under the breaking $SU(8) \to SO(8)$ we have $\mathbf{36} = \mathbf{1} + \mathbf{35}$.) However, to satisfy the D-flatness conditions we must have at least two hypermultiplets in **36** of $U(8)$, so that the aforementioned Higgsing is not possible[44]. This is in accord with the fact that no distribution of D5-branes in the bulk giving rise to the $SO(8)$ gauge symmetry

[43]Note that this is the case even if we place D5-branes (in a fashion compatible with the action of the $\Omega R J'$ orientifold) at $\mathbf{Z}_3$ orbifold fixed points not fixed under R. That is, the corresponding part of the twisted Chan-Paton matrix γ_θ must be traceless to satisfy tadpole cancellation conditions.

[44]One way to see this is as follows. Note that the $U(1)$ subgroup of the $U(8)$ gauge group is anomalous. When all D5-branes are on top of the O5$^+$-plane at the origin of $T^4/\mathbf{Z}_3$, this $U(1)$ factor is broken via the generalized Green-Schwarz mechanism involving the twisted closed string sector hypermultiplet coming from the $\mathbf{Z}_3$ fixed point at the origin. However, if we move D5-branes away from the origin, the $U(1)$ breaking can no longer involve this twisted hypermultiplet - the latter is localized at the corresponding fixed point. In fact, the singlet of $SO(8)$ in the decomposition $\mathbf{36} = \mathbf{1} + \mathbf{35}$ under $SU(8) \to SO(8)$ carries a non-zero $U(1)$ charge (which is equal $+2$ in the aforementioned normalization). This is precisely the singlet whose VEV would measure the separation between the O5$^+$-plane at the origin and the D5-branes in the bulk had the Higgsing been possible. However, in this case this singlet would also have to be the one eaten in Higgsing the $U(1)$ factor, so that we would have only the $SO(8)$ gauge bosons but no massless hypermultiplets coming from D5-branes in the bulk. This, however, would mean that there is no modulus corresponding to the separation between the O5$^+$-plane and D5-branes. We, therefore, conclude that the aforementioned Higgsing is indeed impossible.

could possibly be $\mathbf{Z}_3$ symmetric. Thus, D5-branes are *stuck* at the O5$^+$-plane located at the origin of $T^4/\mathbf{Z}_3$ in this model.

Finally, let us discuss the $b=4$ $\Omega R J'$ orientifold. In this case we have 10 O5$^-$-planes and 6 O5$^+$-branes, and the unique distribution of O5-planes compatible with the $\mathbf{Z}_3$ symmetry is the following: those at the fixed points $(0,0), (e_a/2, d_a/2)$ are of the O5$^-$ type, while those at the fixed points $(0, d_a/2), (e_a/2, 0)$ are of the O5$^+$ type. In this case we have only 8 D5-branes all of which must be placed at the origin, so that the twisted tadpole cancellation condition

$$\mathrm{Tr}(\gamma_\theta) = 8 \tag{171}$$

can be satisfied. Note that the sign in this case is plus instead of minus as in the $b=2$ case as here the O5-plane at which the D5-planes are placed is of the O5$^-$ type. The unique solution to the above tadpole condition is $\gamma_\theta = I_8$, so that the gauge group of this model is $SO(8)$ with no massless hypermultiplets in the open string sector. The closed string sector spectrum is the same as in the $b=2$ case. In fact, the massless spectrum of this model is the same as that of the $\Omega J'$ orientifold with $b=4$. Note that, just as in the $b=2$ case, in the $b=4$ model D5-branes are also stuck at the origin of $T^4/\mathbf{Z}_3$ - there are no massless open string hypermultiplets in this model.

Before we end this subsection let us note that in the case of $\Omega J'$ orientifolds the inability to Higgs the gauge group is interpreted as impossibility of turning on Wilson lines compatible with the $\mathbf{Z}_3$ symmetry in such a way that all tadpoles are canceled.

C. The $\mathbf{Z}_6$ Models

In this subsection we will discuss the $\Omega J'$ orientifolds of Type IIB on $T^4/\mathbf{Z}_6$ with $b=2,4$. As before, we will assume that $T^4 = T^2 \otimes T^2$. Let g be the generator of $\mathbf{Z}_6$. Then, since $\mathbf{Z}_6 \approx \mathbf{Z}_2 \otimes \mathbf{Z}_3$, we can write $g = R\theta$, where R and θ are the generators of the $\mathbf{Z}_2$ and $\mathbf{Z}_3$ subgroups, respectively, with the following actions on the coordinates z_1, z_2 parametrizing the two 2-tori: $Rz_{1,2} = -z_{1,2}$, $\theta z_1 = \omega z_1$, $\theta z_2 = \omega^{-1} z_2$ ($\omega \equiv \exp(2\pi i/3)$). The action of J' on the closed string untwisted as well as R twisted sector fields is trivial, while J' interchanges the θ twisted sector with the θ^{-1} twisted sector, as well as the g twisted sector with the g^{-1} twisted sector.

To begin with let us discuss the closed sting sector of the above orientifold. The untwisted sector gives rise to the six dimensional $\mathcal{N}=1$ supergravity multiplet plus one tensor multiplet and 2 hypermultiplets. The g and g^{-1} twisted sectors together give rise to one tensor multiplet and one hypermultiplet. The θ and θ^{-1} twisted sectors together give rise to 5 tensor multiplets and 5 hypermultiplets. As to the R twisted sector, we must consider $b=0,2,4$ cases separately as the Ω projection acts differently on the corresponding fixed points for different values of b.

In the $b=0$ case before the $\mathbf{Z}_3$ projection in the R twisted sector we have 16 fixed points. At all of these fixed points we have O5$^-$-planes, so that each of them gives rise to a hypermultiplet (but no tensor multiplets). Note that the fixed point at the origin is invariant under the $\mathbf{Z}_3$ twist θ. The other 15 fixed points fall into 5 distinct groups, each group containing 3 fixed points which are permuted by the action of θ. In each of these

5 groups we can form one linear combination of the corresponding 3 fixed points which is invariant under θ. Thus, we have total of 6 hypermultiplets coming from the R twisted sector - one from the origin, and the other 5 from the rest of the fixed points.

Next, let us consider the $b = 2$ case. Here we have 12 $O5^-$-planes and 4 $O5^+$-planes. At the fixed point at the origin we have an $O5^+$-plane. Thus, this fixed point gives rise to a tensor multiplet. The other 3 fixed points at which we have $O5^+$-planes together give rise to another tensor multiplet. Finally, the 12 fixed points at which we have $O5^-$-planes together give rise to 4 hypermultiplets. Thus, the R twisted sector gives rise to 4 hypermultiplets and 2 tensor multiplets in the $b = 2$ case.

In the $b = 4$ case we have 10 $O5^-$-planes and 6 $O5^+$-planes. At the fixed point at the origin we have an $O5^-$-plane. This fixed point, therefore, gives rise to a hypermultiplet. The other 9 fixed points at which we have $O5^-$-planes together give rise to 3 additional hypermultiplets. Finally, the 6 fixed points at which we have $O5^+$-planes together give rise to 2 tensor multiplets. Thus, the R twisted sector gives rise to 4 hypermultiplets and 2 tensor multiplets in the $b = 4$ case, which is the same as in the $b = 2$ case[45].

Next, let us discuss the open string sector in the $b = 2$ model. In this model we have 32 D9-branes and 16 D5-branes. Note that from our discussion of the corresponding $\mathbf{Z}_3$ model it follows that all D5-branes in this models must be placed at the $O5^+$-plane at the origin of $T^4/\mathbf{Z}_6$. This then, following our discussion in subsection A, implies that the twisted Chan-Paton matrix $\gamma_{R,5}$ must have eigenvalues ± 1, and so must the matrix $\gamma_{R,9}$. From this it follows that the Wilson lines in the 99 sector must be of the D_4 type, which is consistent with our discussion in subsection B[46].

Here we can ask whether the $b = 2$ $\mathbf{Z}_6$ model is consistent once we make the aforementioned choices. First, recall that the $b = 2$ $\mathbf{Z}_2$ model with the Wilson lines of the D_4 type (and the twisted Chan-Paton matrices with $\nu^2 = +1$ - see subsection A) suffers from

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[45] This corrects the error in [5], where the R twisted sector in the $b = 2, 4$ $\mathbf{Z}_6$ models was thought to give rise to 6 hypermultiplets and no tensor multiplets.

[46] These points were missed in [5] in the discussion of this model. There it was erroneously assumed that the O5-plane at the origin, at which all D5-branes were placed, is of the $O5^-$ type. Consequently, the matrices $\gamma_{R,5}$ and $\gamma_{R,9}$ were assumed to have eigenvalues $\pm i$, and, in the language we are using here, the Wilson lines in the 99 sector were assumed to be of the D_4' type. From the above discussions it should be clear that such a setup would be inconsistent as it is not even $\mathbf{Z}_3$ symmetric (so that the $\mathbf{Z}_3$ orbifolding procedure would be inconsistent). In fact, it is not difficult to show that with these assumptions the corresponding massless spectrum would have to be anomalous (as some of the tadpoles would not be canceled). The corresponding spectrum given in [5], however, is free of, say, the R^4 anomaly. One of the errors made in [5] that had lead to this seemingly consistent spectrum, as we have already mentioned, was the incorrect computation of the number of twisted tensor multiplets in the closed string spectrum. Another error, related to the multiplicity of states, was made in the discussion of the 59 sector. In fact, this point is rather non-trivial, and we will discuss it in more detail in a moment. Finally, the θ projection was carried out erroneously in [5], which was already noticed in [41]. All these errors added up to give the erroneous spectrum reported in [5].

the presence of half-hypermultiplets in real representations in the 59 sector. The $b = 2$ $\mathbf{Z}_6$ model, therefore, is expected to have a similar problem as well. However, as we have already mentioned, the 99 $\mathbf{Z}_2$ discrete gauge symmetry is broken by the $\mathbf{Z}_3$ twist. This might at first seem to imply that in the $\mathbf{Z}_6$ model unlike the $\mathbf{Z}_2$ model we might be able to avoid the difficulty with the 59 half-hypermultiplets. In fact, if this were so, then this model at first might seem to be consistent even for the conformal field theory orbifold - recall from subsection B that in the $b = 2$ $\Omega R J'$ $\mathbf{Z}_3$ model we cannot move D5-branes away from the O5$^+$-plane at the origin of K3 (and, similarly, in the $b = 2$ $\Omega J'$ $\mathbf{Z}_3$ model we cannot Higgs the 99 gauge group by turning on Wilson lines). This then implies that in the $b = 2$ $\mathbf{Z}_6$ model *a priori* we do not have one of the problems we have encountered in the corresponding $\mathbf{Z}_2$ model, in particular, that related to the inconsistencies arising once we move D5-branes from an O5$^+$-plane to an O5$^-$-plane, or *vice-versa*. However, such a conclusion would immediately run into a puzzle with our discussion of the role of the twisted B-flux - recall that we do *not* expect to be able to consistently have O5$^+$-planes within the conformal field theory orbifold if the orbifold group contains the $\mathbf{Z}_2$ generator R (albeit, O5$^+$-planes are perfectly consistent with conformal field theory orbifolds of odd order). In fact, as we will see in a moment, in the $b = 2$ $\mathbf{Z}_6$ model there is indeed a subtle inconsistency in the 59 sector[47].

Thus, let us understand the 59 sector in this model. In fact, to understand the point we would like to make here it suffices to consider the 59 sector before the $\mathbf{Z}_2$ orbifold projection. We can alternatively view this as introducing D5-brane probes [42] in the $b = 2$ $\Omega J'$ $\mathbf{Z}_3$ model. Note that the 59 sector states are in bifundamental representations of the 55 and 99 gauge groups. In fact, for our purposes here the precise 55 quantum numbers are not going to be relevant. This is related to the fact that in the 55 sector we have just the $\mathbf{Z}_3$ orbifold projection, which is straightforward to carry out. However, in the 99 sector we have the additional projections coming from the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold. Thus, we would like to understand how $\mathbf{32}$ of $SO(32)$ decomposes under the gauge group left unbroken after the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold as well as the $\mathbf{Z}_3$ orbifold projections[48]. Before we do this, however, it is instructive to consider the analogous decomposition for the adjoint of $SO(32)$. Note that the Wilson line $\gamma_{S_1,9}$ breaks $SO(32)$ down to $U(16)$. On the other hand, the twisted Chan-Paton matrix $\gamma_{\theta,9}$ breaks $SO(32)$ down to $SO(16) \otimes U(8)$. Now, the maximal common subgroup of $U(16)$ and $SO(16) \otimes U(8)$ is $U(8) \otimes U(8)$. Under the breaking $SO(32) \supset U(8) \otimes U(8)$ the adjoint of $SO(32)$ decomposes as follows:

$$\begin{aligned}\mathbf{496} = &(\mathbf{64},\mathbf{1})_{1,1} \oplus (\mathbf{1},\mathbf{64})_{1,1} \oplus (\mathbf{8},\overline{\mathbf{8}})_{1,\omega^{-1}} \oplus (\overline{\mathbf{8}},\mathbf{8})_{1,\omega} \oplus \\ &(\mathbf{28},\mathbf{1})_{-1,1} \oplus (\mathbf{1},\mathbf{28})_{-1,1} \oplus (\mathbf{8},\mathbf{8})_{-1,\omega} \oplus \\ &(\overline{\mathbf{28}},\mathbf{1})_{-1,1} \oplus (\mathbf{1},\overline{\mathbf{28}})_{-1,1} \oplus (\overline{\mathbf{8}},\overline{\mathbf{8}})_{-1,\omega^{-1}}\ ,\end{aligned} \tag{172}$$

where the subscript indicates the $\mathbf{Z}_2$ valued phase due to the $\gamma_{S_1,9}$ projection as well as the $\mathbf{Z}_3$ valued phase due to the $\gamma_{\theta,9}$ projection. The states with the $\mathbf{Z}_2$ phase -1 are all heavy,

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[47] It is then not difficult to show that similar conclusions hold for the $b = 4$ $\mathbf{Z}_6$ model as well.

[48] Note that before any of the orbifold projections the 95 sector states are in the following half-hypermultiplet of $SO(32)_{99} \otimes Sp(16)_{55}$: $\frac{1}{2}(\mathbf{32};\mathbf{16})$.

so that the massless states all have the $\mathbf{Z}_2$ phase $+1$. Such states with the $\mathbf{Z}_3$ phase 1 are the gauge bosons of $U(8) \otimes U(8)$, while the states with the $\mathbf{Z}_3$ phases ω and ω^{-1} combine into one massless hypermultiplet in $(\mathbf{8}, \overline{\mathbf{8}})$ of $U(8) \otimes U(8)$.

Next, the gauge group left unbroken after the full $\mathcal{T}$ orbifold projection can be determined as follows. The action of the second Wilson line $\gamma_{S_2,9}$ amounts to permuting the two $U(8)$ subgroups in $U(8) \otimes U(8)$ (left unbroken by $\gamma_{S_1,9}$ and $\gamma_{\theta,9}$) accompanied by the complex conjugation. Thus, for instance, $(\mathbf{8}, \mathbf{1})$ of $U(8) \otimes U(8)$ is mapped to $(\mathbf{1}, \overline{\mathbf{8}})$ by the action of $\gamma_{S_2,9}$. This implies that the final unbroken gauge group is $U(8)$, and the massless matter consists of one hypermultiplet in $\mathbf{36}$ of $U(8)$. Note that normally we would expect the appearance of the $\mathbf{Z}_2$ discrete gauge subgroup in the breaking $U(8) \otimes U(8) \rightarrow U(8)_{\rm diag} \otimes \mathbf{Z}_2$. However, as we will show in a moment, this $\mathbf{Z}_2$ discrete gauge group is actually broken in the case under consideration. Note that it is the 59 sector massless states that are expected to carry non-trivial 99 $\mathbf{Z}_2$ discrete gauge quantum numbers. However, as we will see momentarily, the 59 sector states in this model do not carry well defined gauge quantum numbers at all.

To see this, let us discuss the decomposition of $\mathbf{32}$ of $SO(32)$ under $SO(32) \rightarrow U(8) \otimes U(8) \rightarrow U(8)$. Under the first breaking we have:

$$\mathbf{32} = (\mathbf{8}, \mathbf{1})_1 \oplus (\overline{\mathbf{8}}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{8})_\omega \oplus (\mathbf{1}, \overline{\mathbf{8}})_{\omega^{-1}} \ . \tag{173}$$

Here we have only shown the $\mathbf{Z}_3$ valued phases due to the $\gamma_{\theta,9}$ projection. In fact, we did not give the phases (which are actually $\mathbf{Z}_4$ valued) due to the $\gamma_{S_1,9}$ projection for the reason that the Wilson lines (more precisely, the freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold generators) do *not* act in the 59 sector (just as they do not act in the 55 sector). One way to see this is to note that the Wilson lines can only act on states with Kaluza-Klein momenta (but not windings) coming from the compact directions. This can also be seen by noting that continuously turning on Wilson lines (which is equivalent to Higgsing the 99 gauge group by giving VEVs to the 99 hypermultiplets) in, say, the $b = 0$ $\mathbf{Z}_2$ model does *not* change the number of 59 hypermultiplets. In fact, this is precisely the key reason for the problem we are going to point out next. Note that under the action of $\gamma_{S_2,9}$ the state $(\mathbf{8}, \mathbf{1})_1$ is mapped to the state $(\mathbf{1}, \overline{\mathbf{8}})_{\omega^{-1}}$, and, similarly, the state $(\overline{\mathbf{8}}, \mathbf{1})_1$ is mapped to the state $(\mathbf{1}, \mathbf{8})_\omega$. Thus, the linear combinations that carry well defined gauge quantum numbers under the unbroken $U(8)$ gauge group are given by:

$$|\mathbf{8}\rangle_\pm \equiv \frac{1}{\sqrt{2}} \Big(|(\mathbf{8}, \mathbf{1})_1\rangle \pm |(\mathbf{1}, \overline{\mathbf{8}})_{\omega^{-1}}\rangle \Big) \ , \tag{174}$$

$$|\overline{\mathbf{8}}\rangle_\pm \equiv \frac{1}{\sqrt{2}} \Big(|(\overline{\mathbf{8}}, \mathbf{1})_1\rangle \pm |(\mathbf{1}, \mathbf{8})_\omega\rangle \Big) \ , \tag{175}$$

where the subscript on the left hand side indicates the $\mathbf{Z}_2$ gauge quantum numbers with the $\mathbf{Z}_2$ subgroup arising in the breaking $U(8) \otimes U(8) \rightarrow U(8)_{\rm diag} \otimes \mathbf{Z}_2$. Note, however, that the states $|\mathbf{8}\rangle_\pm$ and $|\overline{\mathbf{8}}\rangle_\pm$ do *not* carry well defined $\mathbf{Z}_3$ quantum numbers. This implies that if in the 59 sector we perform the $\mathbf{Z}_3$ orbifold projection by keeping the $\mathbf{Z}_3$ invariant states, then the latter will not have consistent couplings to the $U(8)$ gauge bosons. If instead we keep the above linear combinations with consistent couplings to the $U(8)$ gauge bosons, then these states will be incompatible with the $\mathbf{Z}_3$ orbifold projection. Either way we have an

inconsistency in the 59 sector of this model, which, in turn, is consistent with our discussions in subsection A[49].

Before we end this subsection, we would like to discuss the massless spectra of the $\mathbf{Z}_6$ models with B-flux. More precisely, here we can discuss the 99 and 55 as well as closed string sectors. The closed string sector in both $b = 2$ and $b = 4$ models contains (together with the six dimensional $\mathcal{N} = 1$ supergravity multiplet) 9 tensor multiplets and 12 hypermultiplets. The gauge group of the $b = 2$ model is $[U(4) \otimes U(4)]_{99} \otimes [U(4) \otimes U(4)]_{55}$. The massless hypermultiplets in the 99 and 55 sectors are given by:

$$(\mathbf{4}, \mathbf{4}; \mathbf{1}, \mathbf{1})_{99} \ , \tag{176}$$

$$(\mathbf{1}, \mathbf{1}; \mathbf{4}, \mathbf{4})_{55} \ . \tag{177}$$

As to the 95 sector, the irreducible R^4 and F^4 anomaly cancellation would require that we have the following massless hypermultiplets:

$$(\mathbf{4}, \mathbf{1}; \mathbf{4}, \mathbf{1})_{95} \ , \tag{178}$$

$$(\mathbf{1}, \mathbf{4}; \mathbf{1}, \mathbf{4})_{95} \ . \tag{179}$$

However, as we discussed above, the 59 sector states in this model do *not* carry well defined gauge quantum numbers, hence inconsistency in this model.

Finally, in the $b = 4$ model the gauge group is $U(4)_{99} \otimes U(4)_{55}$, and there are no massless hypermultiplets in the 99 and 55 sectors[50]. As to the 95 sector, the irreducible R^4 and F^4 anomaly cancellation would require that we have *two* massless hypermultiplets in $(\mathbf{4}; \mathbf{4})_{95}$. Note, however, that as in the $b = 2$ model the 95 sector states in this model do not carry well defined gauge charges. This is due to the fact that the 99 $(\mathbf{Z}_2)^2$ discrete gauge symmetry is completely broken by the $\mathbf{Z}_3$ twist. This, in turn, is consistent with the fact that the 95 sector states do not carry well defined gauge quantum numbers - it would otherwise be difficult to understand how we can have *two* copies of the aforementioned 95 hypermultiplets as there is no discrete gauge symmetry to distinguish the corresponding vertex operators[51].

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[49] Here the following remark is in order. Consider probe D1-branes in the $\Omega J'$ $\mathbf{Z}_3$ models with B-flux. Then in the 19 open string sector we would encounter the same problem as that we have just discussed for D5-brane probes. This might signal that there could be a non-perturbative inconsistency in these $\mathbf{Z}_3$ models, which would have to be visible in the dual heterotic compactification. In particular, on the heterotic side this inconsistency might manifest itself via the world-sheet theory of the fundamental heterotic string being inconsistent. Note, however, that such a problem is absent in the $\Omega R J'$ $\mathbf{Z}_3$ orientifolds with B-flux which could, therefore, be consistent even non-perturbatively. Here we note that the $\Omega J'$ and $\Omega R J'$ $\mathbf{Z}_3$ models with B-flux are different at the massive (which are non-BPS) levels, which might be the reason why one set of these models could have different non-perturbative behavior compared with the other.

[50] This corrects the error in the corresponding spectrum given in [5].

[51] Here we note that before the $\mathbf{Z}_3$ orbifold projection, that is, in the corresponding $b = 4$ $\mathbf{Z}_2$

Thus, we conclude that the $\mathbf{Z}_6$ models with B-flux cannot be made completely consistent within this framework[52].

D. The $\mathbf{Z}_4$ Models

In this section we will discuss the $\Omega J'$ orientifolds of Type IIB on $T^4/\mathbf{Z}_4$ (for simplicity we will assume that $T^4 = T^2 \otimes T^2$) with $b = 2, 4$. Let g be the generator of $\mathbf{Z}_4$. Its action on the complex coordinates z_1, z_2 parametrizing the two 2-tori is given by: $gz_1 = iz_1, gz_2 = -iz_2$. Note that $g^2 \equiv R$ is the generator of the $\mathbf{Z}_2$ subgroup of $\mathbf{Z}_4$ ($Rz_{1,2} = -z_{1,2}$). The action of J' on the closed string untwisted as well as R twisted sector fields is trivial, while J' interchanges the g twisted sector with the g^{-1} twisted sector.

Lets us first discuss the closed string sector of the above orientifold. The untwisted sector gives rise to the six dimensional $\mathcal{N} = 1$ supergravity multiplet plus one tensor multiplet and 2 hypermultiplets. The g and g^{-1} twisted sectors together give rise to 4 tensor multiplets and 4 hypermultiplets. As to the R twisted sector, as in the $\mathbf{Z}_6$ case, we must consider $b = 0, 2, 4$ cases separately.

In the $b = 0$ case we have 16 fixed points in the R twisted sector before the $\mathbf{Z}_4$ projection. At all of these fixed points we have $\mathrm{O5}^-$-planes, so each of them gives rise to a hypermultiplet (but no tensor multiplets). Let e_a and d_a be the vielbeins corresponding to the two 2-tori. Then the 4 fixed points at $(0,0), (e_3/2, 0), (0, d_3/4), (e_3/2, d_3/2)$ are invariant under the $\mathbf{Z}_4$ twist g. The other 12 fixed points fall into 6 distinct groups, each group containing 2 fixed points which are permuted by the action of g (note that g acts as a $\mathbf{Z}_2$ twist on these fixed points which follows from the fact that by definition $g^2 = R$ must be 1 on all points fixed under R). In each of these 6 groups we can form one linear combination of the corresponding 2 fixed points which is invariant under g. Thus, we have total of 10 hypermultiplets coming from the R twisted sector - 4 from the points fixed under both R and g, and 6 from the points fixed under R but not g.

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model we have *four* distinct vertex operators in the 59 sector distinguished by their charges under the 99 $(\mathbf{Z}_2)^2$ discrete symmetry. To have an anomaly free model, however, only *two* of these vertex operators would have to survive the $\mathbf{Z}_3$ projection, while the other two would have to be projected out. Thus, the $\mathbf{Z}_3$ twist would have to act non-trivially on the corresponding quantum numbers. In fact, it indeed does, except that its action is incompatible with the 99 gauge quantum numbers - as we have already explained, the $\mathbf{Z}_3$ invariant states in the 59 sector do not possess well defined gauge quantum numbers. The reason for this is that the $\mathbf{Z}_3$ twist does not commute with the already non-commuting Wilson lines. The situation in the $b = 2$ $\mathbf{Z}_6$ model is analogous to that we have just described for the $b = 4$ model, except that there are additional subtleties due to the fact that before the $\mathbf{Z}_3$ projection we have half-hypermultiplets in the corresponding $\mathbf{Z}_2$ model.

[52]The above massless spectra (with the guessed 95 matter content) are free of the irreducible anomalies, so that there might exist consistent string constructions giving rise to these spectra. However, the $\Omega J'$ orientifolds of Type IIB on $T^4/\mathbf{Z}_6$ with B-flux do not seem to be the corresponding constructions.

Next, let us consider the $b=2$ case. Here we have 12 $O5^-$-planes and 4 $O5^+$-planes. For definiteness let us assume that the B-flux is turned on on the first T^2. Then the $\mathbf{Z}_4$ symmetry requires one of the following two distributions of the O5-planes. We can have 4 $O5^+$-planes at the fixed points $(0,0),(0,d_a/2)$, or we can have 4 $O5^+$-planes at the fixed points $(e_3/2,0),(e_3/2,d_a/2)$. Both setups give equivalent models, so we will focus on the latter setup for definiteness. Note that the fixed points $(e_3/2,0)$ and $(e_3/2,d_3/2)$ are also fixed under g, so that these fixed points give rise to one tensor multiplet each. On the other hand, the two fixed points $(e_3/2,d_1/2)$ and $(e_3/2,d_2/2)$ are permuted by the action of g, so that they together give rise to one tensor multiplet. Finally, it is not difficult to see that the rest of the fixed points (at which we have $O5^-$-planes) give rise to 7 hypermultiplets. Thus, in the $b=2$ case we have 3 tensor multiplets and 7 hypermultiplets coming from the R twisted sector[53].

In the $b=4$ case we have 10 $O5^-$-planes and 6 $O5^+$-planes. One of the four equivalent distributions of O5-planes consistent with the $\mathbf{Z}_4$ symmetry is the following. At the fixed points $(0,0)$ and $(e_a/2,d_b/2)$ we have $O5^-$-planes, while at the fixed points $(0,d_a/2)$ and $(e_a/2,0)$ we have $O5^+$-planes. After the g projection, the latter give rise to 4 tensor multiplets, while the former give rise to 6 hypermultiplets. Thus, in the $b=4$ case we have 4 tensor multiplets and 6 hypermultiplets in the R twisted sector[54].

Next, let us discuss the open string sector. First let us focus on the $b=2$ model. Note that we have the B-flux in the direction of the first T^2, so that we have the corresponding non-commuting Wilson lines in the directions e_1 and e_2. Note that the action of g on the corresponding shifts S_a is given by (here $S_3\equiv S_1S_2$):

$$gS_1g^{-1}=S_2~,\quad gS_2g^{-1}=S_1^{-1}=S_1~,\quad gS_3g^{-1}=S_1^{-1}S_2=S_3~, \tag{180}$$

where we have used the fact that the shifts S_i are $\mathbf{Z}_2$ valued, that is, $S_i^2=1$. This implies that similar relations must also hold for the corresponding Chan-Paton matrices $\gamma_{S_a,9}$ and $\gamma_{g,9}$. There are, however, immaterial sign ambiguities in these relations such as whether to require $\gamma_{g,9}\gamma_{S_3,9}\gamma_{g,9}^{-1}=+\gamma_{S_3,9}$ or $-\gamma_{S_3,9}$. This ambiguity is related to the fact that even though the S_a shifts are $\mathbf{Z}_2$ valued, the corresponding γ_{S_a} matrices can be $\mathbf{Z}_4$ valued in the sense that $\gamma_{S_a}^2$ need not be the identity matrix I_{32} but can also be equal $-I_{32}$ (in the latter case $-\gamma_{S_a}=\gamma_{S_a}^{-1}$). In the following it will be convenient to use the following choices for these signs:

$$\gamma_{g,9}\gamma_{S_1,9}\gamma_{g,9}^{-1}=\gamma_{S_2,9}~,\quad \gamma_{g,9}\gamma_{S_2,9}\gamma_{g,9}^{-1}=\gamma_{S_1,9}~,\quad \gamma_{g,9}\gamma_{S_3,9}\gamma_{g,9}^{-1}=-\gamma_{S_3,9}~. \tag{181}$$

A solution to these conditions can be written in terms of 16 copies of the corresponding 2 dimensional representations:

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[53]This corrects the error in [5], where the R twisted sector in the $b=2$ $\mathbf{Z}_4$ model was thought to give rise to 2 tensor multiplets and 8 hypermultiplets.

[54]This corrects the error in [5], where the R twisted sector in the $b=4$ $\mathbf{Z}_4$ model was thought to give rise to 3 tensor multiplets and 7 hypermultiplets.

$$\gamma_{S_1,9} = \kappa\sigma_3 \otimes I_{16} \ , \tag{182}$$
$$\gamma_{S_2,9} = \kappa\sigma_1 \otimes I_{16} \ , \tag{183}$$
$$\gamma_{S_3,9} = i\sigma_2 \otimes I_{16} \ , \tag{184}$$
$$\gamma_{g,9} = \zeta_g \otimes \Gamma_g \ , \tag{185}$$

where $\kappa^2 = 1$ corresponds to the D_4' type of Wilson lines, while $\kappa^2 = -1$ corresponds to the D_4 type of Wilson lines. Note that both of these choices are *a priori* allowed as (181) implies that

$$\gamma^2_{S_1,9} = \gamma^2_{S_2,9} \ , \tag{186}$$

but does not relate $\gamma^2_{S_3,9}$ to $\gamma^2_{S_i,9}$.

Note that in the above solution for $\gamma_{S_a,9}$ and $\gamma_{g,9}$ we have assumed (for definiteness) that $\eta_{12} = \kappa^2$, albeit this can be relaxed. The 2×2 matrix ζ_g, whose eigenvalues are $1, -1$, is given by:

$$\zeta_g \equiv \frac{1}{\sqrt{2}} \left(\sigma_3 + \sigma_1\right) \ . \tag{187}$$

Finally, the matrix Γ_g is determined as follows. First, the twisted tadpole cancellation conditions imply that $\gamma_{g,9}$ must be traceless [3]. It then follows that Γ_g must be traceless as well. Next, we have

$$\gamma_{R,9} = \gamma^2_{g,9} = I_2 \otimes \Gamma^2_g \ . \tag{188}$$

Note that for $\kappa^2 = +1$ (that is, for the D_4' type of Wilson lines) we must have $\gamma_{R,9}$ (which is also traceless) with eigenvalues $\pm i$, while for $\kappa^2 = -1$ (that is, for the D_4 type of Wilson lines) we must have $\gamma_{R,9}$ with eigenvalues ± 1. This fixes Γ_g (up to equivalent representations) as follows:

$$\kappa^2 = +1 : \quad \Gamma_g = \mathrm{diag}(\alpha, \alpha^{-1}, -\alpha, -\alpha^{-1}) \otimes I_4 \ , \tag{189}$$
$$\kappa^2 = -1 : \quad \Gamma_g = \mathrm{diag}(1, -1, i, -i) \otimes I_4 \ , \tag{190}$$

where $\alpha \equiv \exp(2\pi i/8)$.

The above discussion can be generalized to the $b = 4$ case as well. Here we have two sets of Wilson lines $\gamma_{S_a,9}$ and $\gamma_{T_a,9}$ corresponding to the first and the second T^2, respectively. A solution for the Wilson lines and the twisted Chan-Paton matrix $\gamma_{g,9}$ satisfying all the required consistency conditions is given by:

$$\gamma_{S_1,9} = \kappa\sigma_3 \otimes I_2 \otimes I_8 \ , \tag{191}$$
$$\gamma_{S_2,9} = \kappa\sigma_1 \otimes I_2 \otimes I_8 \ , \tag{192}$$
$$\gamma_{S_3,9} = i\sigma_2 \otimes I_2 \otimes I_8 \ , \tag{193}$$
$$\gamma_{T_1,9} = I_2 \otimes \sigma_3 \otimes I_8 \ , \tag{194}$$
$$\gamma_{T_2,9} = I_2 \otimes \sigma_1 \otimes I_8 \ , \tag{195}$$
$$\gamma_{T_3,9} = I_2 \otimes i\sigma_2 \otimes I_8 \ , \tag{196}$$
$$\gamma_{g,9} = \zeta_g \otimes \zeta_g \otimes \widetilde{\Gamma}_g \ , \tag{197}$$

where the 8×8 matrix $\widetilde{\Gamma}_g$ is given by

$$\kappa^2=+1: \quad \widetilde{\Gamma}_g=\mathrm{diag}(\alpha,\alpha^{-1},-\alpha,-\alpha^{-1})\otimes I_2 \ , \tag{198}$$
$$\kappa^2=-1: \quad \widetilde{\Gamma}_g=\mathrm{diag}(1,-1,i,-i)\otimes I_2 \ . \tag{199}$$

Here we note that in the $b=4$ case for $\kappa^2=+1$ the 99 gauge group before the $\gamma_{g,9}$ projection is $SO(8)\otimes(\mathbf{Z}_2)^2$, while for $\kappa^2=-1$ it is $Sp(8)\otimes(\mathbf{Z}_2)^2$. In the $b=2$ case with the aforementioned choice of Wilson lines the 99 gauge group before the $\gamma_{g,9}$ projection is $SO(16)\otimes\mathbf{Z}_2$ for $\kappa^2=+1$, while for $\kappa^2=-1$ it is $Sp(16)\otimes\mathbf{Z}_2$.

Note that the twisted Chan-Paton matrix $\gamma_{g,5}$ is fixed (up to equivalent representations) once $\gamma_{g,9}$ is fixed. In fact, without loss of generality we can choose it to be given by[55]

$$b=2: \quad \gamma_{g,5}=\Gamma_g \ , \tag{200}$$
$$b=4: \quad \gamma_{g,5}=\widetilde{\Gamma}_g \ . \tag{201}$$

This implies that before the $\gamma_{g,5}$ projection (assuming that we place all D5-branes at the same O5-plane) the 55 gauge group is $SO(32/2^{b/2})$ for $\kappa^2=+1$, and $Sp(32/2^{b/2})$ for $\kappa^2=-1$ ($b=2,4$). That is, in the former case we must place D5-branes at an O5$^-$-plane, while in the latter case we must place D5-branes at an O5$^+$-plane. This is in complete parallel with our discussion of the corresponding $\mathbf{Z}_2$ models.

In fact, here we run into the same problem as in the $\mathbf{Z}_2$ case if we attempt to interpret the above orientifold in the context of the conformal field theory orbifold. More precisely, there are two separate issues here. As we will point out in a moment, in the $\mathbf{Z}_4$ models with B-flux we appear to have a problem with the 59 sector states analogous to that in the $\mathbf{Z}_6$ models. That is, the 59 vertex operators are not well defined. However, the issue that we would like to discuss first depends only on the structure of the 55 (and 99) sector states, whose vertex operators are well defined. So for a moment we will (erroneously) assume that the $\mathbf{Z}_4$ orbifold action yields consistent 59 vertex operators corresponding to a discrete gauge group which is $(\mathbf{Z}_2)^{\otimes(b/2)}$ or a subgroup thereof. With this assumption we can derive the spectra of the corresponding models.

First, let us start with the $b=2$ model with $\kappa^2=+1$. The gauge group is $[U(4)\otimes U(4)\otimes \mathbf{D}]_{99}\otimes[U(4)\otimes U(4)]_{55}$, and the massless hypermultiplets are given by:

$$(\mathbf{6},\mathbf{1};\mathbf{1},\mathbf{1})_{99} \ , \quad (\mathbf{1},\mathbf{6};\mathbf{1},\mathbf{1})_{99} \ , \quad (\mathbf{4},\mathbf{4};\mathbf{1},\mathbf{1})_{99} \ , \tag{202}$$
$$(\mathbf{1},\mathbf{1};\mathbf{6},\mathbf{1})_{55} \ , \quad (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{6})_{55} \ , \quad (\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{4})_{55} \ , \tag{203}$$
$$(\mathbf{4}_D,\mathbf{1};\mathbf{4},\mathbf{1})_{95} \ , \quad (\mathbf{1},\mathbf{4}_D;\mathbf{1},\mathbf{4})_{95} \ , \tag{204}$$

where the subscript D in the 95 sector indicates the 99 $\mathbf{D}$ discrete gauge quantum numbers. Here we encounter the first signs of trouble. In particular, note that for no choice of the discrete gauge symmetry $\mathbf{D}$ can we cancel, say, the R^4 irreducible anomaly (recall that the closed string sector contains 13 hypermultiplets and 8 tensor multiplets in this model).

[55] Here we are assuming that all D5-branes are placed at an O5-plane located at a $\mathbf{Z}_4$ fixed point. As we will show in a moment, other *a priori* allowed configurations would be continuously connected to these ones had the $\mathbf{Z}_4$ models with B-flux been consistent, so this assumption is not restrictive.

Next, consider the $b=4$ model with $\kappa^2=+1$. The gauge group is $[U(2)\otimes U(2)\otimes \mathbf{D}]_{99}\otimes [U(2)\otimes U(2)]_{55}$, and the massless hypermultiplets are given by:

$$(\mathbf{1}_a,\mathbf{1};\mathbf{1},\mathbf{1})_{99}~,\quad (\mathbf{1},\mathbf{1}_a;\mathbf{1},\mathbf{1})_{99}~,\quad (\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})_{99}~, \tag{205}$$
$$(\mathbf{1},\mathbf{1};\mathbf{1}_a,\mathbf{1})_{55}~,\quad (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}_a)_{55}~,\quad (\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})_{55}~, \tag{206}$$
$$(\mathbf{2}_D,\mathbf{1};\mathbf{2},\mathbf{1})_{95}~,\quad (\mathbf{1},\mathbf{2}_D;\mathbf{1},\mathbf{2})_{95}~, \tag{207}$$

where $\mathbf{1}_a$ is the antisymmetric representation of $U(2)$ (it is a singlet of $SU(2)$ but is charged under the $U(1)$ factor). Note that this spectrum, just as in the $b=2$ case, cannot be made free of, say, the R^4 irreducible anomaly (the closed string sector contains 12 hypermultiplets and 9 tensor multiplets in this model).

The gauge group of the $\kappa^2=-1$ model with rank b B-flux is $[U(K)\otimes Sp(K)\otimes Sp(K)\otimes \mathbf{D}]_{99}\otimes[U(K)\otimes Sp(K)\otimes Sp(K)]_{55}$, where $K\equiv 8/2^{b/2}$ ($b=2,4$). The massless hypermultiplets are given by:

$$(\mathbf{K},\mathbf{K},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1})_{99}~,\quad (\mathbf{K},\mathbf{1},\mathbf{K};\mathbf{1},\mathbf{1},\mathbf{1})_{99}~, \tag{208}$$
$$(\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{K},\mathbf{K},\mathbf{1})_{55}~,\quad (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{K},\mathbf{1},\mathbf{K})_{55}~, \tag{209}$$
$$(\mathbf{K}_D,\mathbf{1},\mathbf{1};\mathbf{K},\mathbf{1},\mathbf{1})_{95}~,\quad \frac{1}{2}(\mathbf{1},\mathbf{K}_D,\mathbf{1};\mathbf{1},\mathbf{K},\mathbf{1})_{95}~,\quad \frac{1}{2}(\mathbf{1},\mathbf{1},\mathbf{K}_D;\mathbf{1},\mathbf{1},\mathbf{K})_{95}~. \tag{210}$$

Just as for the $\kappa^2=+1$ models, these spectra cannot be made free of the R^4 anomaly (the closed string matter contents in the $\kappa^2=-1$ models are the same as in the corresponding $\kappa^2=+1$ models). Moreover, the 95 sector spectrum is actually anomalous as it contains half-hypermultiplets in real representations[56]. This is similar to what happens in the corresponding $\mathbf{Z}_2$ models.

In fact, the main reason why we have given the above spectra for both $\kappa^2=\pm1$ models is that, if we assume that they are realized as orientifolds of the conformal field theory $T^4/\mathbf{Z}_4$ orbifold, then they are continuously connected. This is in complete parallel with the corresponding discussion in the $\mathbf{Z}_2$ models. Thus, for instance, in the conformal field theory orbifold setup there is no obstruction to moving D5-branes in, say, the $b=2$ model with $\kappa^2=+1$ from an O5$^-$-plane to an O5$^+$-plane. This then leads to the inconsistency in the 59 sector spectrum where half-hypermultiplets arise in various inappropriate representations. To see that the Higgsing corresponding to such motion of D5-branes is allowed, note that in this model (that is, the $b=2$ model with $\kappa^2=+1$) we can simultaneously turn on VEVs of the hypermultiplets $(\mathbf{1},\mathbf{1};\mathbf{6},\mathbf{1})_{55}$, $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{6})_{55}$ and $(\mathbf{1},\mathbf{1};\mathbf{4},\mathbf{4})_{55}$ in such a way that the D-flatness conditions are satisfied. The maximal unbroken 55 gauge subgroup then is $Sp(4)$. This corresponds to moving together 2 dynamical 5-branes off the O5$^-$-plane. Note that each dynamical 5-brane is made of 8 D5-branes - one pairing is due to the orientifold

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[56]Here we note that the above spectrum differs in the 95 sector from that given for these models in the second reference in [19]. However, neither spectra are completely consistent due to the problem with the 95 vertex operators we are going to point out in a moment. In fact, this problem is at least partially responsible for an ambiguity in determining the 95 sector spectrum in these models, which had led to that reported in the second reference in [19].

projection, while another grouping in multiples of 4 is due to the $\mathbf{Z}_4$ orbifold projection. In this subspace of the moduli space, which corresponds to these two dynamical 5-branes sitting on top of each other in the bulk, we have one 55 hypermultiplet in $\mathbf{1}\oplus\mathbf{5}$ of $Sp(4)_{55}$. In fact, the VEV of the singlet measures the separation between the D5-branes and the O5$^-$-plane. We can split the two dynamical 5-branes from each other by giving a VEV to the hypermultiplet in $\mathbf{5}$ of $Sp(4)$, which breaks the gauge group down to $Sp(2)\otimes Sp(2)$. Thus, each dynamical 5-brane, as expected, gives rise to one $Sp(2)$ subgroup, and the leftover *two* singlet hypermultiplets measure the individual locations of the two dynamical 5-branes in the bulk. Note that once all 16 D5-branes approach an O5$^+$-plane, the 55 gauge group is enhanced to $[U(4)\otimes Sp(4)\otimes Sp(4)]_{55}$. In fact, the matter content of the 55 sector in the $b=2$ model with $\kappa^2=-1$ is just right to Higgs the latter gauge group down to $Sp(4)_{55}$ plus the hypermultiplet in $\mathbf{1}\oplus\mathbf{5}$ (or its $Sp(2)\otimes Sp(2)$ subgroup with two singlet hypermultiplets). This then implies that we will have inconsistencies arising in the 59 sector once we, say, move D5-branes from an O5$^-$-plane to an O5$^+$-plane in the $b=2$ model with $\kappa^2=+1$. In fact, these inconsistencies are of the same type as those we have encountered in the corresponding $\mathbf{Z}_2$ models in subsection A.

The above discussion, just as in the corresponding $\mathbf{Z}_2$ cases, leads us to the conclusion, which is consistent with our discussions in subsection A, that we cannot consistently have O5$^+$-planes in the context of the conformal field theory $T^4/\mathbf{Z}_4$ orbifold. This is not surprising as $\mathbf{Z}_4$ contains a $\mathbf{Z}_2$ subgroup, and we have already come to such a conclusion in the $T^4/\mathbf{Z}_2$ case. In the latter models we also pointed out a possibility of deforming the orbifold by partially blowing-up the collapsed $\mathbf{P}^1$'s and turning off the twisted B-flux at the $\mathbf{Z}_2$ fixed points where we expect the O5$^+$-planes. The resulting K3 compactification could then *a priori* be the correct framework for considering the corresponding orientifolds with B-flux, at least, in the $\mathbf{Z}_2$ case we did not find any obvious inconsistencies with such a possibility. So here we can ask whether a similar possibility exists in the $\mathbf{Z}_4$ cases as well. The answer to this question appears to be negative for at least two reasons. First, the massless spectra of the $\kappa^2=+1$ (as well as $\kappa^2=-1$) models cannot be made anomaly free. We would like to discuss the second reason next.

The point is that, just as in the $\mathbf{Z}_6$ models with B-flux (and for essentially the same reason), the 99 $(\mathbf{Z}_2)^{\otimes(b/2)}$ discrete gauge symmetry is broken by the $\mathbf{Z}_4$ orbifold action. This, in turn, implies that the above spectra, in particular, in the 59 sector, are not completely correct. More concretely, in the 59 sector the states invariant under the $\gamma_{g,9}$ (together with $\gamma_{g,5}$) $\mathbf{Z}_4$ orbifold projection do *not* carry well defined gauge quantum numbers at all. Just as in the $\mathbf{Z}_6$ cases, this is due to the fact that $\gamma_{g,9}$ does not commute with already non-commuting Wilson lines $\gamma_{S_i,9}$.

Let us illustrate this point for the $b=2$ model with $\kappa^2=+1$ (all other cases can be treated similarly). Note that in this case $\gamma_{S_a,9}=\gamma_a\otimes I_{16}$ (where γ_a are the corresponding 2×2 matrices), and $\gamma_{g,9}=\zeta_g\otimes\Gamma_g$. The non-commutativity between $\gamma_{g,9}$ and $\gamma_{S_a,9}$ is in the 2×2 block corresponding to γ_a and ζ_a. So, we can ignore the other 16×16 block structure for our purposes here. Moreover, just as in the $\mathbf{Z}_6$ cases, the important point is how $\mathbf{32}$ of $SO(32)$ (that is, the 99 gauge group before the $\mathbf{Z}_4$ orbifold as well as $\mathbf{Z}_2\otimes\mathbf{Z}_2$ freely acting orbifold projections) is decomposed under the unbroken 99 gauge group - the 55 part of the corresponding vertex operators cannot possibly give any trouble here. So, for simplicity we will discuss the following Chan-Paton matrices: $\gamma_{S_a,9}=\gamma_a\otimes I_{16}$, $\gamma_{g,9}=\zeta_g\otimes I_{16}$. (This

choice does not satisfy the twisted tadpole cancellation conditions, but this is not going to be relevant here.) Moreover, for definiteness we will choose $\kappa = 1$, so that $\gamma_1 = \sigma_3$, $\gamma_2 = \sigma_1$, $\gamma_3 = i\sigma_2$.

The unbroken gauge group with the aforementioned choice of Chan-Paton matrices is $SO(16)$. This gauge group arises as follows. First, $\gamma_{S_1,9}$ breaks $SO(32)$ down to $SO(16) \otimes SO(16)$. Then $\gamma_{S_2,9}$ breaks $SO(16) \otimes SO(16)$ down to $SO(16)_{\rm diag} \otimes \mathbf{Z}_2$. Finally, $\gamma_{g,9}$ breaks the $\mathbf{Z}_2$ subgroup, but does not affect the $SO(16)_{\rm diag}$ subgroup. To understand this, let us see how $\mathbf{32}$ of $SO(32)$ decomposes under these breakings. First, under $\gamma_{S_1,9}$ we have

$$\mathbf{32} = (\mathbf{16}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{16}) \ . \tag{211}$$

Next, under $\gamma_{S_2,9}$ the two $SO(16)$ subgroups are permuted, so that we have

$$\mathbf{32} = \mathbf{16}_+ \oplus \mathbf{16}_- \ , \tag{212}$$

where the subscript indicates the 99 $\mathbf{Z}_2$ discrete gauge charges. The latter can be quantified as follows. Note that the eigenvectors of the matrix $\gamma_1 = \sigma_3$ are 2-component column-vectors $\psi_\pm$ with the components $\psi^\uparrow_+ = 1$, $\psi^\downarrow_+ = 0$, $\psi^\uparrow_- = 0$, $\psi^\downarrow_- = 1$. Note that the vertex operators in (211) for $(\mathbf{16}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{16})$ are proportional to ψ_+ and ψ_-, respectively. However, the eigenvectors of the matrix $\gamma_2 = \sigma_1$ are

$$\chi_\pm = \frac{1}{\sqrt{2}} \left(\psi_+ \pm \psi_-\right) \ . \tag{213}$$

In fact, the vertex operators for $\mathbf{16}_+$ and $\mathbf{16}_-$ in (212) are proportional to χ_+ and χ_-, respectively. Thus, $\chi_\pm$ provide precisely the part of the vertex operators corresponding to the 99 $\mathbf{Z}_2$ discrete gauge quantum numbers. Note, however, that $\chi_\pm$ are *not* eigenvectors of ζ_g as ζ_g and γ_2 do not commute. In particular, we have

$$\zeta_g \chi_\pm = \psi_\pm \ , \tag{214}$$

so that the action of $\gamma_{g,9}$ on the 59 sector states mixes states with different gauge quantum numbers. That is, the 59 states with distinct gauge quantum numbers under the unbroken $SO(16)$ subgroup are incompatible with the $\mathbf{Z}_4$ orbifold projection. On the other hand, the 59 states invariant under the $\mathbf{Z}_4$ orbifold projection do not possess well defined gauge quantum numbers. We therefore conclude that we have an inconsistency at the level of the 59 vertex operators in this and other $\mathbf{Z}_4$ models with B-flux.

E. Summary

Let us briefly summarize the results of this section. In subsection A we have argued that the Ω orientifold of Type IIB on the conformal field theory $T^4/\mathbf{Z}_2$ orbifold with B-flux is inconsistent. This can be seen by noting that in the toroidal orbifold with B-flux we would have to introduce O5$^+$-planes, which cannot be placed at the conformal field theory $T^4/\mathbf{Z}_2$ orbifold fixed points. The obstruction here is due to the twisted half-integer B-flux (which makes the conformal field theory orbifold non-singular) stuck in the corresponding collapsed $\mathbf{P}^1$'s. We have also pointed out that perhaps by deforming the orbifold from the conformal

field theory point (such a deformation would involve blowing up the fixed points where we expect the O5$^+$-planes and turning off the corresponding twisted B-flux) we could make the corresponding orientifold background consistent. Such a K3 compactification, however, would not correspond to an exactly solvable conformal field theory, and it is unclear at present how to make the corresponding statements quantitatively precise. In particular, the moduli space corresponding to the motion of D5-branes as well as that corresponding to turning on continuous Wilson lines in the 99 sector would have to have some funny properties which we do not know how to argue for (or, for that sake, against). Thus, for instance, D5-branes should not be able to come on top of the O5$^+$-planes - this could in principle be a property of compactifications on such a K3 as the corresponding moduli space is no longer flat, but it is unclear whether it really is. If, however, this is indeed the case, then the $\mathbf{Z}_2$ models with B-flux could be consistent in the context of such K3 compactifications if we place D5-branes on top of an O5$^-$-plane or in the bulk with the gauge bundle corresponding to the $\mathbf{Z}_2$ orbifold twist "without vector structure" in the language of [36]. However, attempts to construct models with $\mathbf{Z}_2$ gauge bundles "with vector structure" run into various inconsistencies. In fact, these inconsistencies can be seen in the language of the effective field theory where they manifest themselves via anomalies. In particular, the fact that the conformal field theory $T^4/\mathbf{Z}_2$ orbifold cannot be the correct background for the corresponding orientifold with B-flux can also be seen in the effective field theory language. Thus, for instance, in the conformal field theory orbifold case there is no obstruction to moving D5-branes from an O5$^-$-plane to an O5$^+$-plane, and at the latter points in the moduli space the massless spectra in the 59 sector are anomalous.

In fact, it is always the 59 sector where the trouble shows up. This is, in turn, not an accident. Thus, there is no reason for any inconsistencies to arise in the 99 or 55 sectors (at least perturbatively). However, the 59 sector by definition (since it contains hypermultiplets in the bifundamental representations of the 99 and 55 gauge groups) requires certain vector structure contrary to the lack of vector structure required by the fact that once we turn on the B-flux the generalized second Stieffel-Whitney class is non-vanishing. This appears to be the key reason for the aforementioned inconsistencies arising in these models.

Another (perhaps indirect) hint of this is what we have found in the $\mathbf{Z}_6$ and $\mathbf{Z}_4$ models with B-flux, namely, that the 59 sector states, if properly projected by the orbifold action, do not carry well defined gauge quantum numbers. Even though in the previous subsections we have only used examples to illustrate this point, the corresponding underlying reason can be stated quite generally.

Thus, consider a general setup with Type IIB compactified on T^d, where for definiteness we will choose d to be even. Let us introduce some number N of D9-branes wrapping T^d. Note that to cancel tadpoles we would have to introduce the O9$^-$-plane and set $N = 32$. However, we will not do so here, that is, we will not introduce any orientifold planes at all, and we will let N be arbitrary - the point we would like to make here is independent of the tadpole cancellation requirements. Next, let us introduce some number N' of Dp-branes transverse to T^d. Even though this is not particularly important for our discussion here, we will assume that $9 - p$ is a multiple of 4 so that supersymmetry is not completely broken in this background. Note that in the $9p$ open string sector we have no momenta or windings (but only oscillator excitations) in the directions of T^d.

Next, we would like to turn on non-zero B-flux on T^d. We can do this by considering

a freely acting orbifold whose generators S_i, $i = 1, \ldots, d$, are shifts in the corresponding directions of T^d (and, therefore, they commute). Note that here we can have discrete torsion Δ_{ij} between different S_i generators. In particular, we must have $\Delta_{ii} = +1$, but Δ_{ij} can be equal -1 for $i \neq j$ as long as both S_i and S_j have even orders. The matrix B_{ij} corresponding to the B-field can then be compactly written as

$$B_{ij} = \frac{1}{2\pi i} \ln\left(\Delta_{ij}\right) \ . \tag{215}$$

Note that $B_{ij} \sim B_{ij} + 1$.

The action of the shifts S_i on the 99 sector Chan-Paton charges is described by $N \times N$ matrices $\gamma_{S_i,9}$, and corresponds to turning on Wilson lines. The string consistency requires that

$$\gamma_{S_i,9}\gamma_{S_j,9} = \Delta_{ij}\gamma_{S_j,9}\gamma_{S_i,9} \ , \tag{216}$$

so that if we have non-trivial discrete torsion, then the corresponding Wilson lines are non-commuting. Here we note that the action of the shifts on the *pp* open string sector is trivial. In the 99 sector, however, it breaks the gauge group $U(N)$ to its subgroup G. Suppose now that we have non-trivial discrete torsion between some generators S_i. That is, let the rank b of the matrix B_{ij} be non-zero. Then the rank of the unbroken gauge group $r(G) < N$. In fact, $r(G) = N/2^{b/2}$. The coset $U(N)/G$ is given by the discrete gauge group $D \equiv (\mathbf{Z}_2)^{\otimes(b/2)}$. Let d_α denote the corresponding discrete charges, $\alpha = 1, \ldots, |D|$ ($|D| = 2^{b/2}$). Then it is not difficult to see that the fundamental $\mathbf{N}$ of $U(N)$ decomposes as follows:

$$\mathbf{N} = \bigoplus_{\ell,\alpha} (\mathbf{N}_\ell, d_\alpha) \ , \tag{217}$$

where $\mathbf{N}_\ell$ are the fundamental representations of the corresponding subgroups of $G = \bigotimes_\ell U(N_\ell)$. (Note that $r(G) = \sum_\ell N_\ell$.) On the other hand, the adjoint of $U(N)$ decomposes as follows:

$$\mathbf{Adj} = \mathbf{N} \otimes \overline{\mathbf{N}} = \bigoplus_{\ell,\ell',\alpha,\beta} (\mathbf{N}_\ell \otimes \overline{\mathbf{N}}_{\ell'}, d_\alpha \otimes \overline{d}_\beta) \ . \tag{218}$$

Note that the gauge bosons of the unbroken gauge group G come from the subset with $\ell = \ell'$ and $\beta = \alpha$. This subset reads:

$$\bigoplus_{\ell,\alpha} (\mathbf{Adj}_\ell, d_\alpha \otimes \overline{d}_\alpha) \ . \tag{219}$$

In fact, this subset contains exactly $|D|$ copies of the adjoint of G as $\left|\bigoplus_\alpha d_\alpha \otimes \overline{d}_\alpha\right| = |D|$. Note that in the 99 sector only one of these copies survives the full freely acting orbifold projection with respect to the action of $\gamma_{S_i,9}$. It is then not difficult to see that the remaining states are given by[57]

[57]Thus, for instance, consider the case where $b = d$, and $\Delta_{12} = \Delta_{34} = \ldots = \Delta_{b-1,b} = -1$ with

$$\bigoplus_{\ell}(\mathbf{Adj}_{\ell}, D_{\rm inv}) \ , \tag{220}$$

where the vertex operator corresponding to the discrete gauge charge $D_{\rm inv}$ is given by

$$|D_{\rm inv}\rangle = \frac{1}{\sqrt{D}}\sum_{\alpha}|d_\alpha \otimes \overline{d}_\alpha\rangle \ . \tag{221}$$

In contrast, in the $9p$ sector no states are projected out by the action of this freely acting orbifold - it actually does not act in the $9p$ sector (neither does it act in the pp sector). So instead the fundamental $\mathbf{N}$ of $U(N)$ in the $9p$ sector simply decomposes under the unbroken gauge group as in (217).

Now suppose we orbifold T^d with the B-field turned on by the action of the orbifold point group Γ such that some of the elements of Γ do not commute with the shifts S_i. It is necessary for the consistency of the theory that the shifts S_i and the twists in Γ form a larger orbifold group $\widetilde{\Gamma}$, which is non-Abelian. Let us consider one non-trivial element g (such that $g^2\neq 1$) that does not commute with some of the shifts S_i. Then this implies that the corresponding Chan-Paton matrices $\gamma_{S_i,9}$ and $\gamma_{g,9}$ also do not commute. It is, however, important to note that the action of the twisted Chan-Paton matrix $\gamma_{g,9}$ does not reduce the rank of the unbroken group $G_g\subset G$, that is, $r(G_g)=r(G)$. This follows from the fact that g simply permutes elements of $\widetilde{\Gamma}$ that are pure shifts among each other. In fact, this implies that we can always find $\gamma_{g,9}$ such that the unbroken gauge group $G_g=G$. In this case the action of $\gamma_{g,9}$ is non-trivial only on the discrete gauge quantum numbers[58]. In fact, this action breaks the discrete gauge symmetry D either completely or to its smaller subgroup. Thus, the action of $\gamma_{g,9}$ on the discrete quantum numbers d_α is given by:

$$\gamma_{g,9}:\ d_\alpha\rightarrow\sum_{\alpha'}c_{\alpha\alpha'}d_{\alpha'} \ , \tag{222}$$

where the matrix $c_{\alpha\alpha'}$ is *not* diagonal as $\gamma_{g,9}$ does not commute with the Wilson lines. This then implies that the $9p$ sector states invariant under the g projection cannot have well defined gauge quantum numbers under the unbroken subgroup G. This can be seen by

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all other $\Delta_{ij}=+1$. Then we can group the b independent projections $\gamma_{S_i,9}$ as follows. First, $b/2$ projections $\gamma_{S_1,9},\gamma_{S_3,9},\dots,\gamma_{S_{b-1},9}$ break the original $U(N)$ gauge group down to its subgroup of rank N. Then, the rest of the projections $\gamma_{S_2,9},\gamma_{S_4,9},\dots,\gamma_{S_b,9}$ break this gauge group to its subgroup $G\otimes D$. Note that the number of gauge bosons in (219) is $|G||D|=2^{b/2}|G|$. This corresponds to keeping the states with trivial discrete gauge charges (indeed, the representations $\bigoplus_\alpha d_\alpha\otimes\overline{d}_\alpha$ correspond to precisely such discrete gauge charges). These states are invariant under half of the $\gamma_{S_i,9}$ projections, say, the $\gamma_{S_2,9},\gamma_{S_4,9},\dots,\gamma_{S_b,9}$ projections. On the hand, the number of states in (219) which are also invariant under the remaining projections $\gamma_{S_1,9},\gamma_{S_3,9},\dots,\gamma_{S_{b-1},9}$ is $|G|$. These states are given by (220).

[58]Our discussion here straightforwardly generalizes to the most general case. However, to illustrate the point we would like to make here, it suffices to consider $\gamma_{g,9}$ of the aforementioned type.

noting that the states in (220) (that correspond to the gauge bosons of the unbroken gauge group G) are invariant under the action of g as[59]

$$\gamma_{g,9} : |D_{\rm inv}\rangle \rightarrow \frac{1}{\sqrt{D}} \sum_\alpha \sum_{\beta,\gamma} c_{\alpha\beta} \overline{c}_{\gamma\alpha} |d_\beta \otimes \overline{d}_\gamma\rangle = \frac{1}{\sqrt{D}} \sum_\alpha |d_\alpha \otimes \overline{d}_\alpha\rangle = |D_{\rm inv}\rangle \ , \tag{223}$$

that is, the aforementioned states are actually invariant under the $\gamma_{g,9}$ action, so that they provide the correct basis for the gauge bosons. On the other hand, the states in the $9p$ sector that are invariant under the g projection are in a basis different from that of the G gauge bosons. This can be seen by noting that in the basis where $\gamma_{g,9}$ acts diagonally on $\mathbf{N}$ of $U(N)$ we have:

$$\mathbf{N} = \bigoplus_{\ell,k} (\mathbf{N}_\ell, k) \ , \tag{224}$$

where the vertex operators corresponding to the quantum numbers k are given by:

$$|k\rangle \equiv \sum_\alpha f_{k\alpha} |d_\alpha\rangle \ . \tag{225}$$

Here $f_{k\alpha}$ are the eigenvectors of the matrix $c_{\alpha\alpha'}$:

$$\sum_\alpha f_{k\alpha} c_{\alpha\alpha'} = \lambda_k f_{k\alpha'} \ , \tag{226}$$

where λ_k are the eigenvalues of $c_{\alpha\alpha'}$. It is important to note that for each k $f_{k\alpha} \neq 0$ for at least two different values of α, which follows from the fact that the twist g and the Wilson lines do not commute. Next, consider a massless $9p$ sector state containing $(\mathbf{N}_\ell, k)$. Its scattering with its own conjugate state then will contain $\mathbf{Adj}_\ell$ together with the following vertex operator corresponding to the discrete gauge quantum numbers:

$$|k\rangle \otimes |\overline{k}\rangle = \sum_{\alpha\alpha'} f_{k\alpha} \overline{f}_{k\alpha'} |d_\alpha \otimes \overline{d}_{\alpha'}\rangle \ . \tag{227}$$

Thus, the aforementioned $9p$ sector states would scatter into states, in particular, gauge bosons, containing non-diagonal terms with $|d_\alpha \otimes \overline{d}_{\alpha'}\rangle$ with $\alpha \neq \alpha'$ absent in (220). Thus, we see that the g invariant states in the $9p$ sector indeed do not carry well defined gauge quantum numbers.

The reason why g acts so differently in the 99 and $9p$ sectors is actually very simple. The Chan-Paton part of the $9p$ vertex operators is proportional to

$$V_{9p} \sim \lambda_9 \overline{\lambda}_p \ , \tag{228}$$

while in the 99 sector we have

[59]Here we are using the fact that $\sum_\alpha c_{\alpha\beta} \overline{c}_{\gamma\alpha} = \delta_{\beta\gamma}$, which is the statement that $\gamma_{g,9}\gamma_{g,9}^{-1} = 1$, in particular, when acting on the discrete gauge charges d_α and $\overline{d}_\alpha$. Also, note that Chan-Paton matrices are unitary, so that the action of g on $\overline{\mathbf{N}}$ of $U(N)$ is given by the matrix $\gamma_{g,9}^{-1}$ - see below.

$$V_{99} \sim \lambda_9 \overline{\lambda}_9 \ , \tag{229}$$

where λ_9 and $\overline{\lambda}_9$ correspond to the fundamental and antifundamental of the 99 gauge group $U(N)$, while λ_p and $\overline{\lambda}_p$ correspond to the fundamental and antifundamental of the pp gauge group $U(N')$. Thus, the action of g in the $9p$ sector is given by

$$g: \ \lambda_9 \overline{\lambda}_p \rightarrow \gamma_{g,9} \lambda_9 \overline{\lambda}_p \gamma^{-1}_{g,p} \ , \tag{230}$$

while in the 99 sector we have

$$g: \ \lambda_9 \overline{\lambda}_9 \rightarrow \gamma_{g,9} \lambda_9 \overline{\lambda}_9 \gamma^{-1}_{g,9} \ . \tag{231}$$

Thus, the action of $\gamma_{g,9}$ in the 99 sector is bilinear, while in the $9p$ sector it is linear, and this is precisely the reason why the two actions are incompatible in the presence of non-commuting Wilson lines (which affect the 99 quantum numbers only) as explained above.

The above discussion implies that we indeed have a conflict between the facts that gauge bundles of D9-branes wrapped on tori with B-flux lack vector structure, while the presence of $9p$ sectors with Dp-branes transverse to such tori implies the presence of certain vector structure. In fact, the inconsistencies we have encountered in orientifolds of Type IIB on $T^4/\mathbf{Z}_M$ orbifolds with B-flux simply imply that the corresponding choices of the gauge bundles are not consistent within this framework (albeit consistent choices could be found for the cases without the B-flux). The obstruction to having consistent gauge bundles, once again, is related to the lack of vector structure.

IV. FOUR DIMENSIONAL ORIENTIFOLDS WITH B-FLUX

In this section we will discuss four dimensional orientifolds with B-flux. In particular, we will consider compactifications with $\mathcal{N}=1$ as well as $\mathcal{N}=2$ supersymmetry. More concretely, we will discuss orientifolds of Type IIB on T^6/Γ orbifolds. If Γ is a (non-trivial) subgroup of $SU(2)$, then we have $\mathcal{N}=2$ supersymmetry, and if Γ is a subgroup of $SU(3)$ but not of $SU(2)$, then we have $\mathcal{N}=1$ supersymmetry. In subsection A we will discuss $\mathcal{N}=2$ examples. In subsections B,C,D,E we will discuss $\mathcal{N}=1$ examples with the orbifold groups $\Gamma = \mathbf{Z}_3, \mathbf{Z}_7, \mathbf{Z}_3 \otimes \mathbf{Z}_3, \mathbf{Z}_6$, respectively.

A. The $\mathcal{N}=2$ Models

In this subsection we will discuss orientifolds of Type IIB on $T^2 \otimes$ K3, where K3 $= T^4/\mathbf{Z}_M$ ($M = 2,3,4,6$). *A priori* the B-flux can be turned on either on T^2 or K3 or both. However, we will focus on the cases with B-flux such that we will *not* encounter the difficulties analogous to those we have found in six dimensional orientifolds with B-flux. In particular, the latter will always be either transverse to or inside of the world-volumes of *all* D-branes present in a given model[60].

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[60]If only one type of D-branes is present, which is the case in the $\mathbf{Z}_3$ models, then we can have B-flux simultaneously turned on in the directions inside of as well as transverse to the D-brane world-volumes.

Let us discuss this point in a bit more detail. What we have found in the previous section is that if we simultaneously have D-branes wrapping tori (or, more precisely, orbifolds thereof) and D-branes transverse to such tori, then we run into various subtle inconsistencies. The latter would not, for instance, occur if all D-branes where transverse to B-flux. Similarly, such inconsistencies would also be absent if all D-branes are wrapping such tori. This is precisely the strategy we are going to employ here to construct consistent four dimensional $\mathcal{N}=2$ supersymmetric orientifolds with B-flux. Note that in the case of K3 orientifolds with both D9- and D5-branes we cannot have such a setup. But with three compact complex dimensions this now becomes possible to achieve.

To begin with, we can consider the following setup. Consider the $\Omega J'$ orientifold[61] of Type IIB on $T^2 \otimes \mathrm{K3}$, where $\mathrm{K3} = T^4/\mathbf{Z}_M$ ($M = 2,3,4,6$). For $M = 2,4,6$, where we have both D9- and D5-branes, we will assume that the B-field on K3 is trivial, while we have non-zero B-flux on T^2. In the $M = 3$ case *a priori* we can have B-flux on both T^2 as well as K3.

Let us first consider the models with $M = 2,4,6$. The B-flux on T^2 can be described in terms of the freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold with discrete torsion. Note that the latter now commutes with the $\mathbf{Z}_M$ orbifold action. In fact, the spectra of these models can be obtained by compactifying the corresponding K3 models of [3] with trivial B-flux on T^2 with B-flux[62]. The closed string spectra are given by a straightforward dimensional reduction of the corresponding six dimensional spectra given in [3][63]. As to the open string sector, its massless spectrum is obtained by performing the freely acting $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ orbifold projections. Note that now the Wilson lines act in both 99 as well as 55 sectors. The corresponding Chan-Paton matrices $\gamma_{S_i,9}$ and $\gamma_{S_i,5}$ must be the same (up to equivalent representations). This is necessary for the action of the $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ freely acting orbifold on the 59 sector states, which is given by the matrices $\gamma_{S_i,9}\gamma^{-1}_{S_i,5}$, to be consistent. Moreover, the twisted Chan-Paton matrices $\gamma_{R,9}$ and $\gamma_{R,5}$ (which up to equivalent representations must be the same so that the $\mathbf{Z}_2 \subset \mathbf{Z}_M$ orbifold twist R acts consistently on the 59 sector states) have eigenvalues $\pm i$ (and not ± 1). This follows from the corresponding statement for the six dimensional $\mathbf{Z}_2$ model with trivial B-flux. Note that depending upon whether the Wilson lines on T^2 are of the D_4 or D'_4 type we have different models, which, however, are connected as they belong to different points of the Coulomb branch of the same $\mathcal{N}=2$ gauge theory.

Note that the ranks of both the 99 and 55 gauge groups are equal[64] 8. In fact, both

[61]The action of J' was defined in the previous section. In particular, note that $J' = 1$ in the $\mathbf{Z}_2$ models.

[62]Note that in the open string sector this is equivalent to compactifying a six dimensional gauge theory on a non-commutative T^2.

[63]Note, in particular, that the B-flux on T^2 does *not* affect the number of six dimensional tensor multiplets, which upon the dimensional reduction actually give rise to four dimensional vector multiplets.

[64]Here we assume that the Wilson lines in both the 99 and 55 sectors correspond to the points of

gauge groups contain $\mathbf{Z}_2$ discrete gauge subgroups, but there are no massless states carrying non-trivial $[\mathbf{Z}_2]_{99}$ or $[\mathbf{Z}_2]_{55}$ charges. This, in particular, applies to the 59 sector states as well where the multiplicity of states now is $\xi_{59}=1$. This is because now the Wilson lines *do* act in the 59 sector. In particular, the 59 states carrying non-trivial $[\mathbf{Z}_2]_{99}$ or $[\mathbf{Z}_2]_{55}$ charges are now massive - they are at heavy Kaluza-Klein levels corresponding to the compactification on T^2. Here we note that in the limit of the large volume T^2 the effect of the B-flux is ameliorated. Thus, as was pointed out in [23], in this limit in all three 99, 55 and 59 sectors the Kaluza-Klein states with non-trivial $[\mathbf{Z}_2]_{99}$ and/or $[\mathbf{Z}_2]_{55}$ charges become massless (along with the Kaluza-Klein states with trivial discrete gauge charges) so that the freely acting orbifold action is ameliorated, and the massless six dimensional states arising in this limit are in the representations of the full rank $16+16$ 99 plus 55 gauge symmetry. That is, in this limit we recover the six dimensional K3 orientifolds of [3].

For illustrative purposes let us consider the $\mathbf{Z}_2$ example. The gauge group (at the $\mathbf{Z}_2\otimes\mathbf{Z}_2$ freely acting orbifold points) is $U(8)_{99}\otimes U(8)_{55}$. (Here we drop the $[\mathbf{Z}_2]_{99}\otimes[\mathbf{Z}_2]_{55}$ discrete gauge subgroup as no massless states carry non-trivial charges with respect to the latter.) If the Wilson lines on T^2 are of the D_4 type, then the massless hypermultiplets are given by:

$$2\times(\mathbf{36};\mathbf{1})~,\quad 2\times(\mathbf{1};\mathbf{36})~,\quad (\mathbf{8};\mathbf{8})~. \tag{232}$$

On the other hand, if the Wilson lines on T^2 are of the D_4' type, then the massless hypermultiplets are given by:

$$2\times(\mathbf{28};\mathbf{1})~,\quad 2\times(\mathbf{1};\mathbf{28})~,\quad (\mathbf{8};\mathbf{8})~. \tag{233}$$

As we have already mentioned, these points belong to the same branch of the moduli space corresponding to the Coulomb branch.

In fact, most of the above discussion also applies to the $M=3$ cases, except that here we can also have B-flux on K3, so that the corresponding four dimensional models are obtained by compactifying the six dimensional $\Omega J'$ $\mathbf{Z}_3$ models with $b=0,2,4$ on T^2 with B-flux, and the former are recovered in the large T^2 limit[65].

Another class of orientifolds we can consider here is the following. Let z_1 parametrize T^2, and z_2,z_3 parametrize T^4 in K3 $=T^4/\mathbf{Z}_M$. For simplicity let us assume that $T^4=T^2\otimes T^2$, where z_2,z_3 parametrize these two 2-tori. (To avoid confusion, from now on we will refer to

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the respective moduli spaces which can be described by the freely acting $\mathbf{Z}_2\otimes\mathbf{Z}_2$ orbifold.

[65] Here we can also consider $\Omega RJ'$ $\mathbf{Z}_3$ orientifolds with B-flux turned on both on T^2 and K3. In the limit of large volume T^2 we recover the six dimensional $\Omega RJ'$ $\mathbf{Z}_3$ models (with D5-branes only) discussed in subsection B of section III. Here we note that there is a possibility of a non-perturbative inconsistency in the four dimensional $\Omega J'$ $\mathbf{Z}_3$ models with B-flux turned on on K3 (regardless of the B-flux on T^2). This is in complete parallel with our discussion for the corresponding six dimensional $\Omega J'$ $\mathbf{Z}_3$ models. Note, however, that such a non-perturbative inconsistency is not expected in the four dimensional $\Omega J'$ $\mathbf{Z}_3$ models with B-flux turned on on T^2 only. Neither should it occur in the four dimensional $\Omega RJ'$ $\mathbf{Z}_3$ models with B-flux on K3 (regardless of the B-flux on T^2).

the 2-torus parametrized by z_1 as $\widetilde{T}^2$.) Then we can consider the $\Omega R'J'$ orientifold of Type IIB on $T^2 \otimes (T^4/\mathbf{Z}_M)$, where the action of R' is given by $R'z_1 = -z_1$, $R'z_2 = -z_2$, $R'z_3 = z_3$. For $M = 3$ we then have only D5-branes wrapping the 2-torus in T^4 (or, more precisely, an orbifold thereof) parametrized by z_3. That is, the locations of these D5-branes are given by points in the directions z_1, z_2. These $\mathbf{Z}_3$ models with B-field turned on in various directions are straightforward to analyze along the lines of our previous discussions. We will therefore focus on the models with $M = 2, 4, 6$, where we have two types of D-branes (intersecting at right angles). Thus, we have D5-branes wrapping the T^2 parametrized by z_3. We also have D5$'$-branes wrapping the T^2 parametrized by z_2. This follows from the fact that we have O5-planes whose world-volumes coincide with the set of points fixed under R', and we also have O5$'$-planes whose world-volumes coincide with the set of points fixed under RR', R being the generator of the $\mathbf{Z}_2$ subgroup of $\mathbf{Z}_M$. Let $\widetilde{R}z_1 = -z_1$. There are four points on $\widetilde{T}^2$ fixed under the action of $\widetilde{R}$: $\xi_0 = 0$, $\xi_a = e_a/2$, $a = 1, 2, 3$, where e_i, $i = 1, 2$, are the vielbeins on $\widetilde{T}^2$, and $e_3 \equiv -e_1 - e_2$. Note that we have total of 12 O5$^-$-planes and 4 O5$^+$-planes. Similarly, we have total of 12 O5$'^-$-planes and 4 O5$'^+$-planes. (This follows from the fact that we have half-integer B-flux on $\widetilde{T}^2$ but trivial B-flux on K3.) For definiteness let us assume that 4 O5-planes corresponding to the ξ_0 fixed point are of the O5$^+$ type. Then the other 12 O5-planes corresponding to the fixed points ξ_a are of the O5$^-$ type. What about the O5$'$-planes? It is not difficult to see that with the above assumption the 4 O5$'$-planes corresponding to the fixed point ξ_0 must be of the O5$'^+$ type. Similarly, the other 12 O5$'$-planes corresponding to the fixed points ξ_a must be of the O5$'^-$ type. This can be seen as follows. First note that the consistency of the R projection in the 55$'$ sector requires that $\gamma_{R,5}$ and $\gamma_{R,5'}$ both have either $\pm i$ or ± 1 eigenvalues. Second, arguments similar to those in subsection A of section III imply that if $\gamma_{R,5}$ has eigenvalues ± 1, then the O5$'$-planes corresponding to a given fixed point on $\widetilde{T}^2$ are of the type opposite to that of the corresponding O5-planes. That is, if the latter are, say, of the O5$^-$ type, then the former are of the O5$'^+$ type. On the other hand, if $\gamma_{R,5}$ has eigenvalues $\pm i$, then the O5$'$-planes corresponding to a given fixed point on $\widetilde{T}^2$ are of the same type as the corresponding O5-planes[66]. Since we must have 4 O5$^+$-planes and 12 O5$^-$-planes as well as 4 O5$'^+$-planes and 12 O5$'^-$-planes (which follows from the tadpole cancellation conditions), it is then clear that $\gamma_{R,5}$ must have eigenvalues $\pm i$, and O5- and

[66]In fact, one can check the above statements in the following simple way. Note that if we T-dualize, say, on T^2 parametrized by z_2 (note that there are no subtleties with this T-duality procedure as the B-flux on this T^2 is trivial), then D5-branes turn into D7-branes, while D5$'$-branes turn in to D3-branes. Analogous statements also apply to the corresponding O-planes. Thus, after T-duality transformation we obtain a background with O3- and O7-planes as well as the corresponding D-branes. For this system we can straightforwardly repeat the argument in the beginning of subsection A of section III (which was carried out for the 59 system but is identical for the 37 system), which (after T-dualizing back to the 55$'$ system) leads precisely to the aforementioned conclusions. Equivalently, we can carry out these arguments for the 55$'$ system by noting that in the 55$'$ sector we have 4 Neumann-Dirichlet boundary conditions just as in the 37 sector or 59 sector.

O5′-planes corresponding to a given fixed point on $\widetilde{T}^2$ must be of the same type[67].

The massless spectra of the $\Omega R' J'$ orientifolds are the same as those of the corresponding $\Omega J'$ orientifolds. However, the massive spectra differ. Note, for instance, that in the $\Omega J'$ orientifold we have 32 D9-branes as well as 32 D5-branes. The rank of the 99 and 55 gauge groups, however, is 8, and we have the $[\mathbf{Z}_2]_{99} \otimes [\mathbf{Z}_2]_{55}$ discrete gauge symmetry under which some massive Kaluza-Klein states are charged non-trivially. Such a discrete gauge symmetry is absent in the corresponding $\Omega R' J'$ orientifolds - we have only 16 D5-branes and 16 D5′-branes. Furthermore, if we take the size of $\widetilde{T}^2$ in the $\Omega R' J'$ orientifolds to infinity, we will *not* obtain six dimensional theories - Lorentz invariance in these backgrounds is always broken to that of a four dimensional theory. In particular, as was pointed out in [23], the rank of the gauge group in such models in *not* enhanced in the large $\widetilde{T}^2$ limit - this is, in fact, a direct consequence of the fact that the number of each type of branes is only 16. What happens in the large $\widetilde{T}^2$ limit is that the O-planes corresponding to the fixed points ξ_a on $\widetilde{T}^2$ are removed to infinity and decouple from the remaining O-planes corresponding to the fixed point ξ_0.

Before we end this subsection, we would like to make a remark on the motion of branes in the $\Omega R' J'$ models. In particular, note that the motion of branes in the direction of $\widetilde{T}^2$ corresponds to different points on the Coulomb branch of the $\mathcal{N} = 2$ gauge theory. Suppose that D5-branes and D5′-branes are on top of the respective O-planes corresponding to the *same* fixed point on $\widetilde{T}^2$. Then we have a non-trivial 55′ massless matter content. If we move, say, D5-branes off the corresponding O5-plane while leaving D5′-branes untouched, then this corresponds to Higgsing the 55 sector gauge group, while the 5′5′ gauge group is untouched. Note that the 55′ states in this process become heavy (due to Higgsing). In the brane language this is simply the statement that 55′ strings now cannot have zero length, so that the corresponding states are always heavy. On the other hand, if we move D5- and D5′-branes off the corresponding fixed points together, then the 55′-sector states remain massless. In the gauge theory language this corresponds to a special subspace of the Coulomb branch where the 55′ hypermultiplets remain massless as the mass term coming from the coupling to the 55 Higgs field is precisely cancelled by the mass term coming from the coupling to the 5′5′ Higgs field.

B. The $\mathcal{N} = 1$ $\mathbf{Z}_3$ Models

In this subsection we would like to discuss four dimensional $\mathcal{N} = 1$ orientifolds of Type IIB on $T^6/\mathbf{Z}_3$, where the generator θ of $\mathbf{Z}_3$ acts on the complex coordinates z_I, $I = 1, 2, 3$, parametrizing T^6 as follows: $\theta z_I = \omega z_I$, where $\omega \equiv \exp(2\pi i/3)$. *A priori* we can consider various orientifolds with O9-, O7-, O5- or O3-planes with B-flux turned on inside of and/or transverse to their world-volumes. Since other cases are straightforward to consider along

[67]Note that this statement for the $\Omega R' J'$ orientifolds is the analogue of the statement for the corresponding $\Omega J'$ orientifolds that the Wilson lines in the 99 and 55 sectors must be of the same type - see above for details.

the lines of our previous discussions, here we will focus on the models with O3-planes[68]. Thus, we will discuss $\Omega R' J' (-1)^{F_L}$ orientifolds of Type IIB on $T^6/\mathbf{Z}_3$, where $R' z_I = -z_I$, and the action of J' is analogous[69] to that in the six dimensional orientifolds discussed in subsections B,C,D of section III. In such an orientifold we have $n_{f-} = 32 + 32/2^{b/2}$ $\text{O}3^-$-planes, and $n_{f+} = 32 - 32/2^{b/2}$ $\text{O}3^+$-planes, where b is the rank of the B-flux. Moreover, it is not difficult to show [15] that the twisted tadpole cancellation conditions read:

$$\mathrm{Tr}(\gamma_\theta) = -(-1)^{b/2} \times 4 \ . \tag{234}$$

Note that for the untwisted Chan-Paton matrix we have $\mathrm{Tr}(\gamma_I) = 32/2^{b/2}$.

Let us consider the $b = 2, 4, 6$ cases separately[70].

• For $b = 2$ we have $n_{f-} = 48$ $\text{O}3^-$-planes and $n_{f+} = 16$ $\text{O}3^+$-planes. The $\mathbf{Z}_3$ symmetry requires that the O3-plane at the origin of T^6 be of the $\text{O}3^+$ type. If we place all 16 D3-branes on top of this O3-plane, then the gauge group is (note that $\mathrm{Tr}(\gamma_\theta) = +4$ in this case) $U(4) \otimes Sp(8)$, and the massless open string sector contains chiral supermultiplets in

$$\Phi_s = 3 \times (\mathbf{10}, \mathbf{1}) \ , \quad Q_s = 3 \times (\overline{\mathbf{4}}, \mathbf{8}) \ , \quad s = 1, 2, 3 \ . \tag{235}$$

There is a non-trivial superpotential in this model given by:

$$\mathcal{W} = \epsilon_{ss's''} \Phi_s Q_{s'} Q_{s''} \ . \tag{236}$$

• For $b = 4$ we have $n_{f-} = 40$ $\text{O}3^-$-planes and $n_{f+} = 24$ $\text{O}3^+$-planes. The $\mathbf{Z}_3$ symmetry requires that the O3-plane at the origin of T^6 be of the $\text{O}3^-$ type. If we place all 8 D3-branes on top of this O3-plane, then the gauge group is (note that $\mathrm{Tr}(\gamma_\theta) = -4$ in this case) $U(4)$, and the massless open string sector contains chiral supermultiplets in $\Phi_s = 3 \times \mathbf{6}$. There are no renormalizable couplings in this model.

• For $b = 6$ we have $n_{f-} = 36$ $\text{O}3^-$-planes and $n_{f+} = 28$ $\text{O}3^+$-planes. The $\mathbf{Z}_3$ symmetry

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[68]Here we note that if we have B-flux inside of the world-volumes of Op-planes and the corresponding Dp-branes in this background, then, just as in the corresponding six dimensional $\mathbf{Z}_3$ models, there is a possibility of a non-perturbative inconsistency. If we, however, consider the models with O3-planes, such a non-perturbative inconsistency is not expected to arise.

[69]Note, however, that, unlike the six dimensional cases, in the four dimensional orientifolds where some of the orbifold group elements twist all three complex coordinates z_I, the action of J' (which is trivial in the untwisted sector, while it maps a g twisted sector to its inverse g^{-1} twisted sector) does not seem to have a well defined geometric interpretation [12]. Here we will ignore potential difficulties with such an interpretation (which could be seen [12], for instance, by considering a map [43] of these orientifolds to F-theory [44]), and assume that such an action is well defined in the conformal field theory context.

[70]Here we should point out that these models were originally discussed in [7]. More precisely, the spectra of the $b = 2, 6$ models given in [7] were erroneous as it was not realized there that the consistent orientifold projection in these cases is of the Sp type. This was originally corrected in [15], and more recently in [27].

requires that the O3-plane at the origin of T^6 be of the O3$^+$ type. If we place all 4 D3-branes on top of this O3-plane, then the gauge group is (note that $\mathrm{Tr}(\gamma_\theta) = +4$ in this case) $Sp(4)$, and there are no massless open string sector states in this model.

C. The $\mathcal{N}=1$ $\mathbf{Z}_7$ Models

In this subsection we discuss four dimensional $\mathcal{N}=1$ orientifolds of Type IIB on $T^6/\mathbf{Z}_7$, where the generator g of $\mathbf{Z}_7$ acts on the complex coordinates z_I, $I=1,2,3$, parametrizing T^6 as follows: $gz_1 = \alpha$, $gz_2 = \alpha^2 z_2$, $gz_3 = \alpha^4 z_3$, where $\alpha \equiv \exp(2\pi i/7)$. As in the previous section, let us focus on the cases where we have O3-planes. Thus, consider the $\Omega R' J' (-1)^{F_L}$ orientifold of Type IIB on $T^6/\mathbf{Z}_7$ ($R' z_I = -z_I$). Here we note that the rank of the B-flux can take only two values in this case: $b=0,6$. The reason why is that only for these values of b does the $\mathbf{Z}_7$ orbifold act crystallographically on T^6. Another way of seeing this is as follows. Note that we have $n_{f\mp} = 32 \pm 32/2^{b/2}$ O3$^\mp$-planes for the rank b B-flux. The O3-plane at the origin is invariant under the $\mathbf{Z}_7$ twist g. However, O3-planes at other fixed points of R' must come in groups of 7 such that they are permuted by the $\mathbf{Z}_7$ orbifold within each group. However, for $b=2,4$ neither $n_{f-}-1$ nor $n_{f+}-1$ are divisible by 7. For $b=0,6$ both $n_{f-}-1$ and n_{f+} are divisible by 7, so we conclude that in these cases the O3-plane at the origin is of the O3$^-$ type. In the $b=6$ case we must place all 4 D3-branes at this O3-plane, which is consistent with the twisted tadpole cancellation condition $\mathrm{Tr}(\gamma_g) = +4$ [15]. The gauge group in this case is $SO(4)$ with no massless open string matter.

D. The $\mathcal{N}=1$ $\mathbf{Z}_3 \otimes \mathbf{Z}_3$ Models

In this subsection we would like to discuss four dimensional $\mathcal{N}=1$ orientifolds of Type IIB on $(T^2 \otimes T^2 \otimes T^2)/(\mathbf{Z}_3 \otimes \mathbf{Z}_3)$, where the action of the generators θ and θ' of the two $\mathbf{Z}_3$ subgroups on the complex coordinates z_I, $I=1,2,3$, parametrizing the three 2-tori is as follows ($\omega \equiv \exp(2\pi i/3)$): $\theta z_1 = \omega z_1$, $\theta z_2 = \omega^{-1} z_2$, $\theta z_3 = z_3$, $\theta' z_1 = z_1$, $\theta' z_2 = \omega z_2$, $\theta' z_3 = \omega^{-1} z_3$. Here we will focus on the $\Omega R' J' (-1)^{F_L}$ orientifolds ($R' z_I = -z_I$). Using our previous results it is then no difficult to show that we have the following models[71]:

- $b=2$. The gauge group is $U(4) \otimes U(4)$ with the following massless open string sector chiral multiplets

$$Q = (\overline{\mathbf{4}}, \overline{\mathbf{4}}) \ , \quad R = (\mathbf{4}, \overline{\mathbf{4}}) \ , \quad \Phi = (\mathbf{1}, \mathbf{10}) \ , \tag{237}$$

and the superpotential

$$\mathcal{W} = QR\Phi \ . \tag{238}$$

- $b=4$. The gauge group is $U(4)$, and in the open string sector we have a massless chiral multiplet in $\mathbf{6}$ of $U(4)$. There are no renormalizable couplings in this case.
- $b=6$. The gauge group is $Sp(4)$, and there is no massless matter in the open string sector.

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[71]Solutions to the tadpole cancellation conditions for these models were found in [15].

E. The $\mathcal{N}=1$ $\mathbf{Z}_6$ Models

In this subsection we would like to discuss four dimensional $\mathcal{N}=1$ orientifolds of Type IIB on $T^6/\mathbf{Z}_6$, where the generators θ and R of the $\mathbf{Z}_3$ and $\mathbf{Z}_2$ subgroups of the $\mathbf{Z}_6$ orbifold group act on the complex coordinates z_I, $I=1,2,3$, parametrizing the three 2-tori (we assume that $T^6=T^2\otimes T^2\otimes T^2$) as follows ($\omega\equiv\exp(2\pi i/3)$): $\theta z_I=\omega z_I$, $Rz_1=z_1$, $Rz_{2,3}=-z_{2,3}$. Note that the $\Omega J'$ $\mathbf{Z}_6$ model with trivial B-flux was originally constructed in the second reference in [9]. In [14][72] the following model with B-flux was discussed[73]. Consider the $\Omega J'$ orientifold of Type IIB on $(T^2\otimes T^2\otimes T^2)/\mathbf{Z}_6$ with $b=2$ B-flux turned on on the second or third T^2 only. It should be clear that in this model we are going to have inconsistencies similar to those we have encountered in the six dimensional $\mathbf{Z}_6$ models with B-flux[74]. To avoid these difficulties we can assume that B-flux is turned on *inside* of both D9- and D5-branes present in this case. That is, consider the $\Omega J'$ $\mathbf{Z}_6$ model with $b=2$ B-flux turned on on the first T^2. The corresponding Wilson lines must be of the same type in both 99 and 55 sectors. In fact, to be compatible with the $\mathbf{Z}_3$ orbifold action, they must be of the D_4 (and not D_4') type. The important point here is that the multiplicity of states in the 59 sector (before the $\mathbf{Z}_3$ orbifold projection) is $\xi_{59}=1$ (and not 2, which would be the case in the model of [14] had it been consistent).

Instead of describing the massless spectrum of the above model, we will discuss that of a different model[75] (these two models actually have identical massless spectra albeit their massive spectra are different). Thus, consider the $\Omega R'J'$ orientifold of Type IIB on $(T^2\otimes T^2\otimes T^2)/\mathbf{Z}_6$, where $R'z_{1,2}=-z_{1,2}$, $Rz_3=z_3$. Let us turn on $b=2$ B-flux on the first T^2 parametrized by z_1. Note that in this model we have two types of D-branes (intersecting at right angles). Thus, we have D5-branes wrapping the T^2 parametrized by z_3. We also have D5$'$-branes wrapping the T^2 parametrized by z_2. Just as in the previous subsection, the O5- and O5$'$-planes corresponding to the same fixed point on T^2 parametrized by z_1 are of the same type. Moreover, to be compatible with the $\mathbf{Z}_3$ orbifold action, the 4 O5-planes corresponding to the origin of this T^2 must be of the O5$^+$ type, and, similarly, the 4 O5$'$-planes corresponding to the origin of this T^2 must be of the O5$'^+$ type. The other

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[72]This model, among other four dimensional $\mathcal{N}=1$ models with and without B-flux, was also discussed in [15].

[73]This model was discussed in [14] in the phenomenological context of TeV-scale brane world. For a partial list of other recent developments in these directions, see [45].

[74]Note that the O5-plane at the R fixed point $z_2=z_3=0$ must be of the O5$^+$ type for the corresponding background to be $\mathbf{Z}_3$ symmetric. This would then require that the twisted Chan-Paton matrices $\gamma_{R,5}$ as well as $\gamma_{R,9}$ have eigenvalues ±1. In [14], however, these Chan-Paton matrices were assumed to have eigenvalues $\pm i$.

[75]The reason for this is that in the former model there is a possibility of a non-perturbative inconsistency arising along the lines of our previous discussions, while in the model we are going to discuss next such an inconsistency is not expected to arise.

12 O5-planes are of the $O5^-$ type, and, similarly, the other 12 $O5'$-planes are of the $O5'^-$ type. Let us place all 16 D5-branes and 16 $D5'$-branes at an $O5^+$-plane and an $O5'^+$-plane, respectively. The solution to the twisted tadpole cancellation conditions reads (up to equivalent representations) [15]:

$$\gamma_{\theta,5} = \gamma_{\theta,5'} = \mathrm{diag}(1,1,\omega,\omega^{-1}) \otimes I_4 \ , \tag{239}$$

$$\gamma_{R,5} = \gamma_{R,5'} = I_4 \otimes i\sigma_3 \otimes I_2 \ . \tag{240}$$

The gauge group of this model is $[U(2) \otimes U(2) \otimes U(4)]_{55} \otimes [U(2) \otimes U(2) \otimes U(4)]_{99}$, and the massless open string chiral matter reads:

$$\Phi_{1,2} = 2 \times (\mathbf{3},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1})_{55} \ , \quad \widetilde{\Phi}_{1,2} = 2 \times (\mathbf{1},\overline{\mathbf{3}},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1})_{55} \ , \tag{241}$$

$$P_{1,2} = 2 \times (\overline{\mathbf{2}},\mathbf{1},\overline{\mathbf{4}};\mathbf{1},\mathbf{1},\mathbf{1})_{55} \ , \quad \widetilde{P}_{1,2} = 2 \times (\mathbf{1},\mathbf{2},\mathbf{4};\mathbf{1},\mathbf{1},\mathbf{1})_{55} \ , \tag{242}$$

$$P_3 = (\overline{\mathbf{2}},\mathbf{1},\mathbf{4};\mathbf{1},\mathbf{1},\mathbf{1})_{55} \ , \quad \widetilde{P}_3 = (\mathbf{1},\mathbf{2},\overline{\mathbf{4}};\mathbf{1},\mathbf{1},\mathbf{1})_{55} \ , \quad R = (\mathbf{2},\overline{\mathbf{2}},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{1})_{55} \ , \tag{243}$$

$$\Phi'_{1,2} = 2 \times (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1},\mathbf{1})_{5'5'} \ , \quad \widetilde{\Phi}'_{1,2} = 2 \times (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\overline{\mathbf{3}},\mathbf{1})_{5'5'} \ , \tag{244}$$

$$P'_{1,2} = 2 \times (\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{2}},\mathbf{1},\overline{\mathbf{4}})_{5'5'} \ , \quad \widetilde{P}'_{1,2} = 2 \times (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2},\mathbf{4})_{5'5'} \ , \tag{245}$$

$$P'_3 = (\mathbf{1},\mathbf{1},\mathbf{1};\overline{\mathbf{2}},\mathbf{1},\mathbf{4})_{5'5'} \ , \quad \widetilde{P}'_3 = (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{2},\overline{\mathbf{4}})_{5'5'} \ , \quad R' = (\mathbf{1},\mathbf{1},\mathbf{1};\mathbf{2},\overline{\mathbf{2}},\mathbf{1})_{5'5'} \ , \tag{246}$$

$$S = (\mathbf{2},\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{1},\mathbf{1})_{55'} \ , \quad T = (\mathbf{1},\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{1},\mathbf{4})_{55'} \ , \quad U = (\mathbf{1},\mathbf{1},\mathbf{4};\mathbf{1},\mathbf{2},\mathbf{1})_{55'} \ , \tag{247}$$

$$\widetilde{S} = (\mathbf{1},\overline{\mathbf{2}},\mathbf{1};\mathbf{1},\overline{\mathbf{2}},\mathbf{1})_{55'} \ , \quad \widetilde{T} = (\overline{\mathbf{2}},\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1},\overline{\mathbf{4}})_{55'} \ , \quad \widetilde{U} = (\mathbf{1},\mathbf{1},\overline{\mathbf{4}};\overline{\mathbf{2}},\mathbf{1},\mathbf{1})_{55'} \ , \tag{248}$$

where $\mathbf{2}$ and $\overline{\mathbf{2}}$ of $U(2)$ carry the $U(1)$ charges $+1$ and -1, respectively, while $\mathbf{3}$ and $\overline{\mathbf{3}}$ of $U(2)$ carry the $U(1)$ charges $+2$ and -2, respectively. Similarly, $\mathbf{4}$ and $\overline{\mathbf{4}}$ of $U(4)$ carry the $U(1)$ charges $+1$ and -1, respectively. Note that this spectrum is the same as that in [14] except for the $55'$ sector multiplicity of states, which in the above model is 1, while in [14] it was assumed to be 2 (in the corresponding 59 sector)[76].

The above model is expected to be consistent. The superpotential in this model is given by:

$$\begin{aligned}\mathcal{W} = {} & P_1\widetilde{P}_2R + P_2\widetilde{P}_1R + \Phi_1P_2P_3 + \Phi_2P_1P_3 + \widetilde{\Phi}_1\widetilde{P}_2\widetilde{P}_3 + \widetilde{\Phi}_2\widetilde{P}_1\widetilde{P}_3 + \\ & P'_1\widetilde{P}'_2R' + P'_2\widetilde{P}'_1R' + \Phi'_1P'_2P'_3 + \Phi'_2P'_1P'_3 + \widetilde{\Phi}'_1\widetilde{P}'_2\widetilde{P}'_3 + \widetilde{\Phi}'_2\widetilde{P}'_1\widetilde{P}'_3 + \\ & S\widetilde{U}P_3 + U\widetilde{S}\widetilde{P}_3 + T\widetilde{T}R + S\widetilde{T}P'_3 + T\widetilde{S}\widetilde{P}'_3 + U\widetilde{U}R' \ .\end{aligned} \tag{249}$$

Using this superpotential it is not difficult to see that if we Higgs the gauge group along the lines of [14][77], then the number of remaining chiral generations for the Pati-Salam gauge group is 2 (and not 3 as it was originally intended in [14]), which is due to the fact that the

[76]Here we should point out that the aforementioned multiplicity being 1 is consistent with the $U(1)$ anomaly cancellation in the above model via the generalized Green-Schwarz mechanism, while attempts to implement the latter with the aforementioned multiplicity being 2 (as in [14]) appear to run into various difficulties [46].

[77]In particular, assume that the S and $\widetilde{S}$ fields acquire non-zero VEVs, which break the gauge group down to $U(2)_{\mathrm{diag}} \otimes U(2)_{\mathrm{diag}} \otimes U(4)_{55} \otimes U(4)_{5'5'}$. Then the observation of [14] is that one

multiplicity of the $55'$ states in the above model is only 1 (and not 2 as was assumed in [14] for the 59 sector states).

Before we end this subsection, let us note that in orientifolds of the above $\mathbf{Z}_6$ orbifold compactification we cannot turn on $b = 4, 6$ B-flux without running into the aforementioned inconsistencies - for $b = 4, 6$ we cannot avoid having two sets of D-branes such that B-flux is inside of the world-volumes of one set of D-branes while the other set is transverse to it[78].

V. COMMENTS

In this section we would like to comment on various issues related to discussions in the previous sections. To begin with, let us note that non-trivial multiplicity of states, which is due to non-trivial B-flux, in the 59 sectors of some of the models discussed in the previous sections appears to be in conflict with $\mathbf{Z}_M$ orbifold projections (with $M = 4, 6$) acting on the corresponding coordinates (that is, those transverse to D5-branes). This observation might be relevant for other compactifications with such non-trivial multiplicity of states in pp' sectors. In particular, such sectors arise in some of the models recently discussed in [48], where the orientifold action involves complex conjugation of the compact coordinates. The analogue of the quantized B-flux in such backgrounds is given by the components of the complex structure corresponding to the B-flux in the Kähler structure under the interchange of the complex and Kähler structures. In fact, the origin of non-trivial multiplicity of states in such orientifolds is also analogous to that in orientifolds with non-zero B-flux, so *a priori* there might be a possibility of subtle inconsistencies, similar to those we have found in the latter backgrounds, also arising in the former ones. It would be interesting to understand this issue in a bit more detail, but this would be outside of the scope of this paper.

The second comment concerns possible generalizations to non-supersymmetric backgrounds with B-flux. Recently compact non-supersymmetric orientifolds have been constructed in, for instance, [49,50] by introducing both D-branes as well as anti-D-branes in orientifolds of Type IIB on orbifolds that preserve some number of supersymmetries. However, regardless of the B-flux such models have one peculiar feature that some of the NS-NS tadpoles must be non-vanishing. The reason for this is that in the aforementioned construction one introduces only O^{--}- and/or O^{++}-planes (see section II for notations) which are supersymmetric in the sense that the NS-NS and R-R tadpoles for each of these O-planes

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can treat, say, the $U(2)_{\mathrm{diag}} \otimes U(2)_{\mathrm{diag}} \otimes U(4)_{55}$ part of this gauge group as the Pati-Salam gauge symmetry. Here we should point out that there are certain subtleties related to the $U(1)$ factors in the aforementioned Higgsing, which we will not discuss in this paper as the key point here is that even if such Higgsing is possible, the number of chiral Pati-Salam generations still cannot be 3.

[78]Note that, for essentially the same reasons, the $\mathcal{N} = 1$ $\mathbf{Z}_2 \otimes \mathbf{Z}_2$ as well as $\mathbf{Z}_2 \otimes \mathbf{Z}_2 \otimes \mathbf{Z}_3$ models with B-flux discussed in [15] also suffer from various subtle inconsistencies for all three values of $b = 2, 4, 6$. Note, however, that the $\mathbf{Z}_2 \otimes \mathbf{Z}_2 \otimes \mathbf{Z}_3$ model with trivial B-flux originally constructed in the first reference in [13] is consistent. This model when interpreted in the phenomenological context has three chiral generations, and its phenomenological implications were studied in [47].

are identical. However, if we now introduce both D-branes and anti-D-branes, then it is clear that we cannot cancel both NS-NS and R-R tadpoles simultaneously. Thus, a D-brane has R-R charge $+1$, while an anti-D-brane has R-R charge -1. Both of these objects, however, give rise to identical NS-NS tadpoles. Thus, if we require R-R tadpole cancellation, then we have some uncanceled NS-NS tadpoles. In [49] the standard argument was employed that such tadpoles might not be dangerous as they could possibly be dealt with via the Fischler-Susskind mechanism. However, *a priori* it is unclear what the corresponding consistent backgrounds would be if any[79].

Actually, we can understand this point in a bit more detail. If some of the twisted NS-NS tadpoles are non-zero, then we would have to employ the Fischler-Susskind mechanism for twisted (as well as untwisted) scalar fields that couple to D-branes and O-planes. Shifting the VEVs of the twisted scalars, however, implies that we blow up the orbifold, and the consistent background is no longer an exactly solvable conformal field theory. Orientifolds of such backgrounds are difficult to study, so it is unclear what the resulting theory would look like if there at all exists the corresponding consistent background. One things, however, is quite clear - *a priori* there is no reason to believe that, if such a non-orbifold background indeed exists, the open string spectrum of the theory would be the same.

There is, however, a way to avoid the aforementioned difficulty by considering backgrounds where all twisted NS-NS tadpoles cancel. Nonetheless, we have some uncanceled untwisted NS-NS tadpoles, which imply that VEVs of some of the untwisted closed string scalars must be shifted. This, in turn, gives us a hint of what the corresponding consistent backgrounds could look like. Thus, imagine that we have Dp-branes as well as D$\overline{p}$-branes transverse to some compact coordinates. To avoid appearance of open string tachyons, we must place Dp-branes and D$\overline{p}$-branes far enough apart from each other. In fact, to avoid a possibility of brane-anti-brane annihilation, we can assume that branes and anti-branes are stuck at, say, the corresponding orbifold fixed points (for definiteness we will assume that at the orbifold fixed point located at the origin we have branes and not anti-branes). Then tachyons are absent if the separation between the fixed points (related to the compactification radii) is large enough. However, supersymmetry is broken, and the cosmological constant is non-zero. In fact, it depends upon the compactification radii. For large enough values of the latter the vacuum energy monotonically decreases with radii. In fact, in the decompactification limit the branes and anti-branes decouple[80] from each other, and the resulting theory with branes located at the origin is now supersymmetric. In fact, all tadpoles in this theory (that is, both R-R and NS-NS tadpoles) cancel. Thus, here we see that there exists a consistent background for such theories which can be reached via the Fischler-Susskind mechanism - it is a (partially) decompactified background with branes only, which is supersymmetric. The latter has lower vacuum energy compared with the original compactified theory with *runaway* scalar potential (as a function of compactification radii).

[79]This is analogous to the situation in the non-supersymmetric $Sp(32)$ theory discussed in section II. In fact, some of the aforementioned backgrounds could be thought of as compactifications of this theory.

[80]This is the case if the number of decompactified dimensions is larger than 2.

In this respect such compactifications are similar to the standard Scherk-Schwarz type of compactifications (or generalizations thereof) which are unstable to decompactification and eventually end up in a supersymmetric vacuum.

Note that we can construct non-supersymmetric theories with both $\mathrm{O}^{--}/\mathrm{O}^{++}$ (that is, supersymmetric) as well as $\mathrm{O}^{-+}/\mathrm{O}^{+-}$ (that is, non-supersymmetric) orientifold planes, where all R-R as well as NS-NS tadpoles cancel. Non-compact versions of such theories (which contain tachyons) were originally constructed in [51][81]. One can easily generalize this construction to compact cases where one can avoid tachyons by considering (partially) freely acting orbifolds. However, as usual, the vacuum energy in such models has a runaway behavior with the stable supersymmetric vacuum reached in a (partial) decompactification limit.

Finally, we note that some of the discussions of non-perturbative K3 orientifolds with B-flux in the third reference in [19] appear to be affected by the results of this paper, in particular, we expect various aforementioned subtleties arising in these models as well, so that they should also be revisited. This, however, is outside of the scope of this paper, and a more detailed discussion of such and related compactifications will be given elsewhere[82]. Also, some other $\mathcal{N}=1$ models (such as the $\mathbf{Z}_6'$ models discussed in the first reference in [19]) should also be considered in this context.

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[81]There such models were discussed in the context of large N gauge theories, so that the presence of closed string tachyons did not pose a problem [52].

[82]In certain (limited) cases non-perturbative orientifolds of [19] can be described in terms of perturbative orientifolds of non-geometric backgrounds. The latter approach makes it possible to better understand the former, including the issues related to non-zero B-flux.

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